An Analysis of VaR-based Capital Requirements*

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Abstract

We study the dynamic investment and reporting problem of a financial institution subject to capital requirements based on self-reported VaR estimates, as in the Basel Committee's Internal Models Approach (IMA), an issue so far unexplored in the banking literature. With constant price coefficients, we show that optimal portfolios display a local three-fund separation property. VaR-based capital requirements induce financial institutions to tilt their portfolios towards assets with high expected return (and high systematic risk), but result nevertheless in a decrease of the overall risk of trading portfolios and of the probability of default. Overall, we find that capital requirements determined on the basis of the IMA can be very effective not only in curbing portfolio risk but also in inducing truthful revelation of this risk. A comparison with capital requirements determined according to the U.S. FED's Pre-Commitment Approach (PCA) is also provided.

Journal of Economic Literature Classification Numbers: D91, D92, G11, C61.

Keywords: Capital requirements, Basel Capital Accord, Internal Models Approach, Precommitment Approach, VaR, portfolio constraints.

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Abstract

We study the dynamic investment and reporting problem of a financial institution subject to capital requirements based on self-reported VaR estimates, as in the Basel Committee's Internal Models Approach (IMA), an issue so far unexplored in the banking literature. With constant price coefficients, we show that optimal portfolios display a local three-fund separation property. VaR-based capital requirements induce financial institutions to tilt their portfolios towards assets with high expected return (and high systematic risk), but result nevertheless in a decrease of the overall risk of trading portfolios and of the probability of default. Overall, we find that capital requirements determined on the basis of the IMA can be very effective not only in curbing portfolio risk but also in inducing truthful revelation of this risk. A comparison with capital requirements determined according to the U.S. FED's Pre-Commitment Approach (PCA) is also provided.

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Financial institutions are required by regulators to maintain minimum levels of capital. This regulation is normally justified as a response to the negative externalities arising from bank failures and to the risk-shifting incentives created by deposit insurance.[1] The 1988 Basel Capital Accord imposed uniform capital requirements based on risk-adjusted assets, defined as the sum of asset positions multiplied by asset-specific risk weights. These risk weights were intended to reflect primarily the credit risk associated with a given asset. In 1996 the Accord was amended to include additional minimum capital reserves to cover market risk, defined as the risk arising from movements in the market prices of trading positions (Basel Committee on Banking Supervision, 1996a).

The 1996 Amendment's Internal Models Approach (IMA) determines capital requirements on the basis of the output of the financial institutions' internal risk measurement systems. Financial institutions are required to report daily their Value-at-Risk (VaR) at the 99% confidence level over a one-day horizon and over a two-week horizon (ten trading days).[2] The minimum capital requirement on a given day is then equal to the sum of a charge to cover "general market risk" and a charge to cover "credit risk" (or idiosyncratic risk), where the market-risk charge is equal to a multiple of the average reported two-week VaR's in the last 60 trading days[3] and the credit-risk charge is equal to 8% of risk-adjusted assets. U.S.-regulated banks and OTC derivatives dealers are subject to capital requirements determined on the basis of the IMA.

The reliance on the financial institution's self-reported VaRs to determine capital requirements creates an adverse selection and moral hazard problem, since the institution has an incentive to under-report its true VaR in order to reduce capital requirements. The procedure suggested by the Basel Committee to address this problem relies on "backtesting" (Basel Committee on Banking Supervision, 1996c): regulators should evaluate on a quarterly basis the frequency of "exceptions" (that is, the frequency of daily losses exceeding the reported VaR) for every financial institution in the most recent twelve-month period and the multiplier used to determine the market risk charge should be increased (according to a given scale varying between 3 and 4) if the frequency of exceptions is high.[4] Additional

[1] See Berger, Herring and Szegö (1995), Freixas and Santomero (2002) or Santos (2002) for a review of the theoretical justifications for bank capital requirements.

[2] Simply stated, VaR is the maximum loss of a trading portfolio over a given horizon, at a given confidence level (i.e., a quantile of the projected profit/loss distribution at the given horizon). To avoid a duplication of risk-measurement systems, financial institutions are allowed to derive their two-week VaR measure by scaling up the daily VaR by the square root of ten (see: Basel Committee on Banking Supervision, 1996b, p. 4).

[3] More precisely, the market-risk charge is equal to the larger of: (i) the average reported two-week VaR's in the last 60 trading days times a multiplier and (ii) the last-reported two-week VaR. However, since the multiplier is not less than 3 (see below), the average of the reported VaR's in the last 60 trading days times a multiplier typically exceeds the last-reported VaR.

[4] The reason why backtesting is based on a daily VaR measure in spite of the fact that the market risk charge is based on a two-week VaR measure is that VaR measures are typically computed ignoring portfolio revisions over the VaR horizon. According to the Basel Committee, "it is often argued that value-at-risk measures cannot be compared against actual trading outcomes, since the actual outcomes will inevitably be 'contaminated' by changes in portfolio composition during the holding period. [...] This argument is persuasive with regard to the use of value-at-risk measures based on price shocks calibrated to longer holding periods. That is, comparing the ten-day, 99th percentile risk measures from the internal models capital requirement with actual ten-day trading outcomes would probably not be a meaningful exercise. In particular, in any given ten day period, significant changes in portfolio composition relative to the initial

corrective actions in response to a high number of exceptions are left to the discretion of regulators.

This paper studies the optimal behavior of a financial institution subject to capital requirements determined according to the IMA. We view the system of capital requirements put in place by the 1996 Amendment as a revelation mechanism designed to induce financial institutions to truthfully reveal the risk (VaR) of their trading portfolios and to support this risk with adequate levels of capital, a view consistent with Rochet (1999) and Jorion (2001, p. 65). Accordingly, we consider the simultaneous optimal choice of a reporting and investment strategy. Since the incentives to truthful revelation arise in part from the threat of increased capital requirements in the future (through an increase in the reserve multiplier), we consider a fully dynamic model with discrete reporting and continuous trading.

Specifically, we consider a financial institution with preferences represented by a risk-averse utility function defined over the market value of its equity capital at the end of the planning horizon.[5] We assume that the institution has fully-insured deposits and limited liability: thus, our model allows for the risk-shifting incentives created by deposit insurance and the option to default. The planning horizon is divided into non-overlapping "backtesting periods", each of which is in turn divided into non-overlapping "reporting periods". The institution is required to continuously maintain its capital above a minimum level, which equals the sum of a charge to cover market risk and a charge to cover credit risk. At the beginning of each reporting period, the institution must report to regulators its claimed VaR as well as the actual loss over the previous period. The market-risk charge for the current reporting period is then equal to a multiple k of the reported VaR,[6] while the credit-risk charge is equal to the sum of asset positions multiplied by asset-specific credit-risk weights.[7] At the end of each backtesting period, the number of exceptions (i.e., the number of reporting periods in which the actual loss exceeded the reported VaR) is computed and this determines the multiple k over the next backtesting period, according to an increasing scale.[8] To capture the cost of any additional regulatory action that might be undertaken in

positions are common at major trading institutions. For this reason, *the backtesting framework described here involves the use of risk measures calibrated to a one-day holding period.*" (Basel Committee on Banking Supervision (1996c, p. 3).

[5]Risk aversion on the part of financial institutions can be justified by, among other things, the value of the institution's charter: see Keeley (1990).

[6]Consistently with empirical evidence, we find that in our model reported VaR's display little variation from one reporting period to the next. Thus, making capital requirements proportional to an average of recently-reported VaR's (rather than proportional to the last-reported VaR) would make little difference in our results.

[7]This is consistent with the Basel Capital Accord, which sets the charge to cover credit risk equal to 0.08 times the sum of asset positions multiplied by asset-specific weights ranging from 0 to 1.5 (see: Basel Committee on Banking Supervision (2001)). This corresponds to risk weights between 0 and $0.08 \times 1.5 = 0.12$ in our definition. Unrated corporate claims (including equity) are assigned a weight of 100% (0.08 in our definition).

[8]Thus, we assume that the VaR measure used for backtesting coincides with that used to determine the capital charge for market risk. This is without loss of generality, as any difference between the two VaR measures can be captured by rescaling the multiplier k. For example, in our numerical calibration we take the reporting period to be one day, as suggested by the Basel Committee, and assume that the multiplier is determined according to the scale suggested by the Basel Committee times the square root of ten: this adjustment captures the fact that the multiplier should be applied to a two-week (rather than one-day) VaR (see footnote 2).

response to losses exceeding the reported VaR, we also allow for the possibility of pecuniary sanctions at the end of any reporting period in which an exception is observed. These sanctions are assumed to be proportional to the amount by which the actual loss exceeds the reported VaR.

Therefore, the financial institution chooses the level of VaR to report in each period by trading off the cost of higher capital requirements in the current period resulting from a higher reported VaR against the benefit of a lower probability of pecuniary sanctions and a lower probability of higher capital requirements in the future as a result of a loss exceeding the reported VaR. In addition, the institution simultaneously chooses a continuously-rebalanced trading strategy for its portfolio, subject to the applicable capital requirements. We stress that the problem we consider differs from a standard investment problem with portfolio constraints, since capital requirements are not exogenously fixed, but vary endogenously as a result of the institution's optimal reporting strategy.

We explicitly characterize the solution of the problem described above using martingale duality (as in Cuoco (1997)) and parametric quadratic programming. Even with constant price coefficients, optimal portfolios in the presence of capital requirements do not display two-fund separation: as capital requirements become progressively more binding following losses, financial institutions find it optimal to rebalance their portfolios in favor of assets characterized by high risk-weight-adjusted expected returns (high systematic risks). However, we show that optimal portfolios satisfy a local three-fund separation property, with the three funds being the riskless asset, the mean-variance efficient portfolio of risky assets and a risk-weight-constrained minimum-variance portfolio of risky assets. For *no* choice of the parameters we find VaR-based capital requirements leading to an increase in overall portfolio risk or to a higher probability of extreme losses and default. In fact, we find that VaR-based capital requirements are effective in completely offsetting the risk-taking incentives generated by deposit insurance and the associated default option, with the risk-taking by a financial institution in the presence of deposit insurance and capital requirements never exceeding that of a similar institution with unlimited liability and no capital requirements.

In general, financial institutions may optimally under-report or over-report their true VaR's, depending on their risk aversion, the current reserve multiplier, the number of exceptions recorded in the current backtesting period, the time remaining to the end of the current backesting period and the level of the pecuniary penalties associated with an exception. Overall, we find that capital requirements determined on the basis of the IMA can be very effective not only in curbing portfolio risks, but also in inducing truthful revelation of these risks. For relative risk aversion coefficients of 0.25 or higher, the threat of a pecuniary sanction equal to 1% of the amount by which the end-of-period loss exceeds the reported VaR is sufficient to ensure that financial institutions never find it optimal to report VaRs that are below true 90% VaRs.

The optimal behavior of a financial institution subject to capital requirements determined in accordance to the IMA has been so far unexplored in the banking literature.

In a static mean-variance framework, Kahane (1977) and Koehn and Santomero (1980) showed that a more stringent capital requirement (in the form of a lower upper bound on feasible leverage ratios) may induce financial institutions to substitute riskier assets for less risky ones and thus may increase the risk of trading portfolios and the probability of default. Kim and Santomero (1988) established that the same result applies to capital requirements determined on the basis of risk-weighted assets, unless the risk weights happen to be proportional to the assets' betas. The conclusion that capital requirements could lead to an increase in risk taking and hence in the likelihood of bank failures has been the subject of extensive discussion in the subsequent literature.[9] Furlong and Keeley (1989) and Keeley and Furlong (1990) argued that the mean-variance framework is inappropriate to analyze the effect of capital requirements in the presence of deposit insurance and limited liability, because limited liability results in skewed equity return distributions. In particular, Furlong and Keeley (1989) considered a value-maximizing financial institution and showed that stricter leverage limits unambiguously reduce optimal risk-taking. The main reason is that such an institution would always choose the portfolio having the maximum possible risk under the capital requirement (i.e., a corner solution) in order to maximize the value of the deposit insurance (a put option). Gennotte and Pyle (1991) extended the analysis of Furlong and Keeley to allow for investment opportunities having non-zero net present value (NPV) and showed that in this setting tighter capital restrictions can lead financial institutions to increase asset risk. However, this result was obtained under the key assumption that the institution finds it optimal to invest in risky assets having *negative* NPV: thus, as long as the institution has access to traded securities (zero-NPV investments) with the same or higher risk, the result in Furlong and Keeley would still hold. While all of the above studies considered a static setting, Blum (1999) used a two-period model to show that the incentives to increase the risk of trading portfolios in response to tighter capital requirements are even higher in a dynamic setting. This is because capital requirements increase the marginal utility of a unit of capital tomorrow and thus can lead to an increase in risk in an effort to increase expected return. Differently from these papers, we take into account the impact of capital requirement on the risk-shifting incentives created by deposit insurance and limited liability in a dynamic setting with continuous trading and hence with portfolio return distributions that are not restricted to be normals or truncated normals. In addition, we allow for more general capital requirements that include a charge for market risk in addition to that based on risk-weighted assets. While asset substitution incentives are present in our model, we find that capital requirements never have perverse effects on risk-taking or on the probability of failure. Moreover, differently from Furlong and Keeley (1989), this lack of perverse effects is not due to the fact that the financial institution acts as a risk-lover and is always at a corner solution.

Sentana (2001), Emmer, Klüppelberg and Korn (2001), Vorst (2001), Basak and Shapiro (2001) and Cuoco, He and Issaenko (2002) considered the investment problem of a trader subject to an exogenous limit on the VaR of the trading portfolio. None of these papers incorporates limited liability or a realistic model of capital requirements. The first four papers considered the case of a fixed VaR limit, which does not capture the constraint imposed by capital requirements on financial institutions. Cuoco, He and Issaenko considered the case in which the limit varies as a function of the value of the trading portfolio. Their results for the proportional case imply that if the VaR of a financial institution's trading portfolio were perfectly and continuously observable by regulators and minimum capital requirements at any given point in time were simply equal to a fixed multiple of the contemporaneous VaR (with no penalties for observed exceptions), then, under the assumption of CRRA pref-

[9] See Jackson (1999) for a review of the related empirical evidence.

erences and unlimited liability, the optimal portfolio for a financial institution subject to capital requirements would involve a constant proportional allocation to the mean-variance efficient portfolio. Moreover, the capital requirement would either always bind or never bind. Neither of these conclusions holds for the more realistic model of capital regulation considered in this paper.[10]

Again in a static setting, Chan, Greenbaum and Thakor (1992) and Giammarino, Lewis and Sappington (1993) studied the optimal design of a mechanism that induces truthful risk revelation in a setting in which regulators also provide deposit insurance. By contrast, our focus is not on mechanism design but on the analysis of the specific mechanism implemented by the 1996 Amendment. Ju and Pearson (1999) examined, in a static setting, the bias that arises when the VaR of a portfolio is determined on the basis of the delta-normal method with variances and covariances estimated using past data: in this case, a trader subject to a binding VaR constraint and possessing private information about the relation between current variances and covariances and historical ones, is able to select portfolios whose true VaR exceeds the estimated VaR and hence to assume risks in excess of the stated limit. Ju and Pearson quantified the extent of this bias assuming that the supervisor monitoring the limit provides no incentives for the trader to reveal his information and that the trader has one of three objectives: maximizing the portfolio VaR, maximizing the portfolio expected return, or minimizing the variance of the difference between the return of the chosen portfolio and the return of an exogenously-given reference portfolio. By contrast, because of the penalties associated with exceptions, the 1996 Amendment does provide incentives to financial institutions to reveal private information about risk: these incentives (in addition to a more realistic investment objective) are important features of the model we consider.

The rest of the paper is organized as follows. Section I describes our model in detail. Section II characterizes optimal trading strategies in the absence of capital requirements. Section III describes our solution approach to the joint reporting and investment problem in the presence of capital requirements and provides some explicit characterization of optimal trading strategies in this section. Section IV provides a numerical analysis. Section V concludes. Appendix A examines capital requirements determined on the basis of the Pre-Commitment Approach (PCA), an alternative approach proposed by the U.S. Federal Reserve Bank.[11] Appendix B contains all the proofs. Appendix C provides some additional auxiliary results.

1 The Model

We consider a financial institution with a planning horizon equal to T backtesting periods, where T is a positive integer. Without loss of generality, we normalize the length of a backtesting period to 1. Each backtesting period comprises n non-overlapping reporting

[10] Leippold, Trojani and Vanini (2001), using asymptotic approximation techniques, extended the analysis of an exogenous proportional VaR limit in Cuoco, He and Issaenko to incorporate stochastically-varying price coefficients and also examine the equilibrium implications of such a limit.

[11] As will be shown in Appendix A, PCA-based capital requirements are a special case of the model of capital requirements studied in Section III. In related work, Kupiec and O'Brien (1997) provide an analysis of PCA-based capital requirements in a static setting.

periods of equal length $\tau = 1/n$. At the beginning of each reporting period, the financial institution is required to report to a regulator its current VaR as well as the actual profit/loss over the previous reporting period. As explained later, the reported VaR determines the capital charge to cover market risk for the period.

The financial institution has liabilities represented by deposits and (equity) capital. For simplicity, we assume that the face value of deposits D is fixed over the planning horizon and that there are no equity issues or dividend payments over this period. Deposits are fully insured and earn the risk-free interest rate, which is paid out continuously to depositors. The market value of deposits is therefore constant and equal to D.

The investment opportunities are represented by $m+1$ long-lived assets. The first asset is riskless and earns a constant continuously-compounded interest rate $r \geq 0$. The other m assets are risky and their price process S (inclusive of reinvested dividends) follows a geometric Brownian motion with drift vector $r\bar{1} + \mu$ and diffusion matrix σ, i.e.,

$$S(t) = S(0) + \int_0^t I^S(s)(r\bar{1} + \mu)\, ds + \int_0^t I^S(s)\sigma\, dw(s),$$

where $I^S(t)$ denotes the $m \times m$ diagonal matrix with elements $S(t)$, $\bar{1} = (1, \ldots, 1)^\top$ and w is an m-dimensional Brownian motion. We assume without loss of generality that σ has rank m.[12] The financial institution can trade continuously and without frictions over $[0, T]$.[13,14]

Letting θ be the m-dimensional stochastic process representing the (dollar) investment in the risky assets, the evolution of the value A of the institution's asset portfolio over any reporting period is then given by

$$dA(t) = (A(t)r + \theta(t)^\top \mu)\ dt + \theta(t)^\top \sigma\, dw(t) - rD\, dt, \tag{1}$$

where the last term reflects interest payments to depositors.

We define the institution's *regulatory capital* $K = A - D$ as the difference between the value of the institution's asset portfolio and the value of the institution's deposits.[15] Since

[12] If $d = \text{rank}(\sigma) < m$, some stocks are redundant and can be omitted from the analysis. Moreover, w can be redefined in this case to be a d-dimensional Brownian motion.

[13] The assumption of continuous frictionless trading is of course a simplification in the case of a financial institution for which loans constitute a significant portion of investments. However, incorporating illiquidity into the present model would significantly add to its complexity. We view the frictionless case as a reasonable starting point for a first analysis of VaR-based capital requirements, especially in view of the increasing use of loan securitization by financial institutions.

[14] While we do not explicitly impose short-sale constraints, our results would be unchanged by these constraints. As it will be shown in Proposition 6, capital requirements never induce financial institutions to short (long) assets that are held long (short) in the unconstrained mean-variance efficient (MVE) portfolio. Thus, assets that are held short in the unconstrained MVE portfolio would never be held in the presence of short-sale constraints (with or without capital requirements) and thus can simply be ignored. On the other hand, assets that are held long in the unconstrained MVE portfolio are also held in non-negative amounts in the presence of capital requirements and thus are unaffected by short-sale constraints.

[15] Our definition of regulatory capital is different from the market value of the institution's equity because, consistently with practice, the market value of deposit insurance is not included in the value of the institution's assets. Thus, at the terminal date T, the market value of the institution's equity is equal to $K(T)^+ = \max[0, K(T)]$. As it will become clear from equations (2) and (3), we assume that capital requirements are defined in terms of regulatory capital, but that the institution has preferences defined in terms of the market value of capital.

the market value of deposits is fixed and there are no new equity issues, it follows from equation (1) that the institution's regulatory capital satisfies

$$dK(t) = \left(K(t)r + \theta(t)^\top \mu\right) dt + \theta(t)^\top \sigma \, dw(t).$$

The financial institution is required to maintain at all times its regulatory capital above a minimum level equal to the sum of the charge to cover general market risk plus a charge to cover credit (or idiosyncratic) risk. The capital charge to cover market risk equals the VaR reported at the beginning of the current reporting period times a multiplier k. The capital charge to cover credit risk equals the sum of the institution's trading positions (long and short) multiplied by asset-specific risk weights. Thus, letting $\beta \in [0,1]^m$ denote the vector of asset risk weights,[16] the capital charge to cover credit risk at time t equals $\beta^\top(\theta(t)^+ + \theta(t)^-)$, where for any $x \in \mathbb{R}$ $x^+ = \max[0, x]$ and $x^- = \max[0, -x]$.[17] Hence, if $VaR \geq 0$ denotes the VaR reported to regulators at the beginning of the current reporting period and k is the currently-applicable multiplier, the institution must satisfy the constraint

$$K(t) \geq k\, VaR + \beta^\top(\theta(t)^+ + \theta(t)^-) \tag{2}$$

at all times during the reporting period.[18]

The institution is subject to pecuniary sanctions at the end of each reporting period in which the actual loss exceeds the reported VaR.[19] We assume that the sanction is proportional to the amount by which the actual loss exceeds the VaR and denote the proportionality coefficient by λ, where $\lambda \geq 0$. At the end of each backtesting period, the number i ($i = 0, 1, \ldots, n$) of reporting periods in which the actual loss exceeded the reported VaR is computed, and the capital reserve multiplier k for the next backtesting period is set equal to $k(i)$, for some given positive numbers $k(0) \leq k(1) \leq \ldots \leq k(n)$.

We assume that the financial institution's trading strategy, and hence the financial institution's true VaR, are unobservable by the regulator. Therefore, the reported VaR can differ from the true VaR. However, the threat of pecuniary sanctions at the end of each reporting period and the revision of the capital reserve multiplier k at the end of each backtesting period represent incentives to not under-report the VaR, while the capital requirement provides an incentive to not over-report.

The financial institution has limited liability and preferences represented by a power HARA utility function over the market value of its capital at the end of the planning horizon. Thus, it chooses a reporting and trading strategy over $[0, T]$ so as to maximize

$$\mathrm{E}\left[u\left(K(T)\right)\right],$$

[16]It is a straightforward extension to allow β to be different also across long and short positions.

[17]Consistently with existing regulation, we are implicitly assuming a zero risk weight for investment in the money market account.

[18]The constraint $K(t) \geq \beta^\top(\theta(t)^+ + \theta(t)^-)$ is identical to the one that would arise in the presence of margin requirements if the trader were allowed to earn market interest on the margin: see Cuoco and Liu (2000). Thus, while we assume that trading is frictionless, margin requirements could be easily accommodated and would amount to an increase in the vector of risk weights β by an amount equal to the proportional margin requirement.

[19]These sanctions are meant to capture reputation costs or additional disciplinary actions that might be undertaken by regulators. The threat of these sanctions is necessary for institutions to optimally report non-zero VaRs.

where

$$u(K) = \frac{(K^+ + \varepsilon)^{1-\gamma}}{1-\gamma} \tag{3}$$

for some $\varepsilon \geq 0$ and $\gamma > 0$, $\gamma \neq 1$.[20]

2 The Unconstrained Problem

We start by deriving an analytical solution for the investment problem of a financial institution not subject to capital requirements:

$$\begin{aligned} V^U(K,t) &= \max_{\theta} \mathrm{E}\left[u\left(K(T)\right) \,\Big|\, K(t) = K\right] \\ \text{s.t.} \quad dK(s) &= \left(K(s)r + \theta(s)^\top \mu\right) ds + \theta(s)^\top \sigma \, dw(s) \\ K(s) &\geq -D \quad \text{for all } s \in [t,T]. \end{aligned} \tag{4}$$

The constraint $K \geq -D$ reflects the fact that the value of the financial institution's assets, which equals $D + K$, cannot be negative. We refer to the problem in (4) as the unconstrained problem. To simplify comparison with the next section,which deals with the constrained problem with capital requirements, we use a martingale duality approach to solve the above problem.

Letting

$$\xi(t) = \exp\left(-\left(r + \frac{|\kappa|^2}{2}\right)t - \kappa^\top w(t)\right) \tag{5}$$

denote the state-price density process, where

$$\kappa = \sigma^{-1}\mu,$$

it follows from Cox and Huang (1989), Pliska (1986) or Karatzas, Lehoczky and Shreve (1987) that the problem in (4) can be equivalently written as the static problem

$$V^U(K,t) = \min_{\psi \geq 0} \max_{x \geq -D} \mathrm{E}_t\left[u(x) - \psi\left(\xi(T)x - \xi(t)K\right)\right], \tag{6}$$

where ψ is a Lagrangian multiplier. Setting $Z = \psi\xi$, it then follows from (5) that

$$dZ(t) = -Z(t)\left(r\,dt + \kappa^\top dw(t)\right)$$

and (6) implies

$$V^U(K,t) = \min_{z \geq 0}\left[\tilde{V}^U(z,t) + zK\right],$$

where

$$\tilde{V}^U(z,t) = \mathrm{E}\left[\tilde{v}^U(Z(T)) \mid Z(t) = z\right] \tag{7}$$

and

$$\tilde{v}^U(z) = \max_{x \geq -D}\left[u(x) - zx\right]. \tag{8}$$

The next proposition describes the solution of the maximization problem in (8). For convenience, let $b = 1 - 1/\gamma$.

[20] The case $\gamma = 1$ can be treated similarly.

Lemma 1 *If $u(0) > -\infty$ (that is, if either $\varepsilon > 0$ or $\gamma < 1$), there exists a unique $z_U \in (0, \varepsilon^{-\gamma}]$ satisfying*

$$-\frac{z_U^b}{b} + (\varepsilon - D)z_U - \frac{\varepsilon^{1-\gamma}}{1-\gamma} = 0. \tag{9}$$

Lemma 2 *Let z_U be the constant in Lemma 1 if $u(0) > -\infty$ and let $z_U = +\infty$ otherwise. Then the solution of the maximization problem in (8) is given by $K = f^U(z)$, where*

$$f^U(z) = \begin{cases} z^{-\frac{1}{\gamma}} - \varepsilon & \text{if } z \le z_U \\ -D & \text{otherwise.} \end{cases} \tag{10}$$

Hence

$$\tilde{v}^U(z) = u(f^U(z)) - zf^U(z) = \begin{cases} -\frac{z^b}{b} + \varepsilon z & \text{if } z \le z_U \\ \frac{\varepsilon^{1-\gamma}}{1-\gamma} + Dz & \text{otherwise.} \end{cases} \tag{11}$$

The function $f^U(z)$ in (10) is decreasing for $z \in (0, +\infty)$ and strictly positive and strictly decreasing for $z \in (0, z_U)$, with a single discontinuity at $z = z_U$. The function $\tilde{v}^U(z)$ in (11) is continuous and convex.

The result in Lemma 2 allows the derivation of an explicit expression for the dual value function in (7) and for the optimal trading strategy.

Proposition 1 *The dual value function $\tilde{V}^U$ in (7) is given by*

$$\begin{aligned} \tilde{V}^U(z,t) \quad = \quad & -\tfrac{z^b}{b} e^{-b\left(r+\frac{1}{2}(1-b)\kappa^2\right)(T-t)} N\left(\tfrac{\log\left(\frac{z_U}{z}\right)+\left(r+\left(\frac{1}{2}-b\right)|\kappa|^2\right)(T-t)}{|\kappa|\sqrt{T-t}}\right) \\ & + \varepsilon z e^{-r(T-t)} N\left(\tfrac{\log\left(\frac{z_U}{z}\right)+\left(r-\frac{1}{2}|\kappa|^2\right)(T-t)}{|\kappa|\sqrt{T-t}}\right) \\ & + \tfrac{\varepsilon^{1-\gamma}}{1-\gamma} N\left(-\tfrac{\log\left(\frac{z_U}{z}\right)+\left(r+\frac{1}{2}|\kappa|^2\right)(T-t)}{|\kappa|\sqrt{T-t}}\right) \\ & + Dze^{-r(T-t)} N\left(-\tfrac{\log\left(\frac{z_U}{z}\right)+\left(r-\frac{1}{2}|\kappa|^2\right)(T-t)}{|\kappa|\sqrt{T-t}}\right), \end{aligned} \tag{12}$$

where N is the standard normal distribution function. In particular, $\tilde{V}^U$ is strictly convex in z for all $t \in [0, T)$.

Proposition 2 *The optimal trading strategy θ^U for the unconstrained problem in (4) is given by $\theta(t) = \theta^U(Z(t), t)$, where*

$$\theta^U(z,t) = z\tilde{V}^U_{zz}(z,t)\left(\sigma\sigma^\top\right)^{-1}\mu. \tag{13}$$

Moreover, under the optimal investment policy, the financial institution's capital at time t is given by $K(t) = K^U(Z(t), t)$, where

$$K^U(z,t) = -\tilde{V}^U_z(Z(t), t).$$

In particular,

$$K^U(T) = f^U(Z(T)),$$

where f^U is the function in (10).

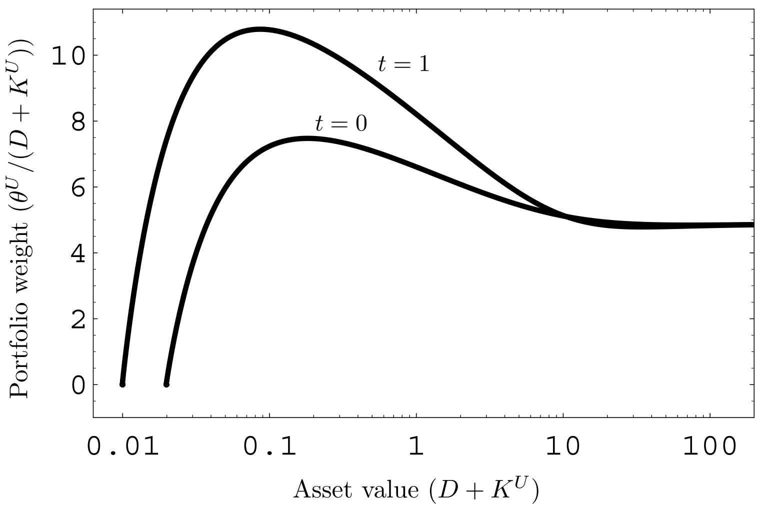

Figure 1: The graph plots the optimal portfolio allocation to stocks in the unconstrained model at $t = 0$ and $t = 1$ as a function of asset value, for the case $\gamma = .25$, $\varepsilon = 0$, $T = 2$, $r = .01$, $\kappa = .27$, $\sigma = .22$ and $D = 1$.

Corollary 1 *We have*

$$\lim_{z \to 0} \theta^U(z,t) = \frac{1}{\gamma} \left(\sigma\sigma^\top\right)^{-1} \mu K^U(z,t),$$

$$\lim_{z \to +\infty} \theta^U(z,t) = 0,$$

$$\lim_{z \to 0} K^U(z,t) = +\infty,$$

$$\lim_{z \to +\infty} K^U(z,t) = \begin{cases} -\, e^{-r(T-t)} D & \textit{if } u(0) > -\infty, \\ 0 & \textit{otherwise.} \end{cases}$$

Proposition 2 implies that, in the absence of capital requirements, the financial institutions has incentives to exploit the option to default by choosing a discontinuous distribution of capital at the terminal date. As long as the state variable $Z(T)$ is lower than z_U, terminal capital equals $Z(T)^{-\frac{1}{\gamma}} - \varepsilon > 0$ and default is avoided. However, in states in which $Z(T)$ is higher than z_U, terminal capital equals $-\,D$: in other words, the institution completely depletes its assets and defaults for the largest possible amount. This is true even in the case $\varepsilon = 0$ as long as the utility function is finite at zero (that is, as long as $\gamma < 1$): infinite marginal utility at zero is not sufficient to prevent default if the institution can default for a non-infinitesimal amount.

The incentives to exploit the default option are also evident in Figure 1, which plots the optimal portfolio weight $\theta^U/(D + K^U)$ as a function of asset value at two different points in time ($t = 0$ and $t = 1$) for the case $\gamma = .25$, $\varepsilon = 0$, $T = 2$, $r = .01$, $\kappa = .27$, $\sigma = .22$ and $D = 1$.[21] As asset value decreases below the default point $e^{-r(T-t)}D$, the proportional

[21] The figure plots asset value on a logarithmic scale, since the portfolio weight decreases steeply to zero as asset value approaches zero.

allocation to risky assets in the absence of capital requirements increases significantly and would become unboundedly large as the end of the investment horizon approaches.

3 The Constrained Problem

3.1 Recursion for the Value Function

Turning next to the institution's investment and reporting problem in the presence of capital requirement, let $V(K, K^-, VaR, i, k, t)$ denote the value function for the institution's problem at time t conditional on current capital being K, capital at the beginning of the current reporting period being K^-, the VaR reported at the beginning of the current reporting period being VaR, the number of exceptions in the current reporting period being i and the current capital reserve multiplier being k. Without loss of generality, suppose that t is in the h-th reporting period, i.e., that $t \in [(h-1)\tau, h\tau)$. Finally, let $\mathcal{T} = \{1, 2, \dots, T\}$ denote the set of backtesting dates. Then it follows from the principle of dynamic programming that

$$V(K, K^-, VaR, i, k, t) = \max_{\theta} \mathrm{E}\left[v(K(h\tau), K^-, VaR, i, k, h\tau) \;\middle|\; K(t) = K \right] \tag{14}$$

$$\text{s.t.} \qquad dK(s) = (K(s)r + \theta(s)^\top \mu)\, ds + \theta(s)^\top \sigma\, dw(s),$$
$$K(s) \geq k\, VaR + \beta^\top (\theta(s)^+ + \theta(s)^-) \quad \text{for all } s \in [t, h\tau),$$

for $K \geq k\, VaR$, where

$$\begin{aligned} & v(K, K^-, VaR, i, k, h\tau) \\ = & \max_{VaR_1 \geq 0} V(K_1, K_1, VaR_1, i_1, k_1, h\tau) 1_{\{K \geq K^- - VaR\}} \\ & + \max_{VaR_2 \geq 0} V(K_2, K_2, VaR_2, i_2, k_2, h\tau) 1_{\{K < K^- - VaR\}} \end{aligned} \tag{15}$$

and

$$K_1 = K, \qquad K_2 = (K - \lambda(K^- - VaR - K))^+,$$

$$i_1 = \begin{cases} 0 & \text{if } h\tau \in \mathcal{T}, \\ i & \text{otherwise}, \end{cases} \qquad i_2 = \begin{cases} 0 & \text{if } h\tau \in \mathcal{T}, \\ i+1 & \text{otherwise}, \end{cases}$$

$$k_1 = \begin{cases} k(i) & \text{if } h\tau \in \mathcal{T}, \\ k & \text{otherwise}, \end{cases} \qquad k_2 = \begin{cases} k(i+1) & \text{if } h\tau \in \mathcal{T}, \\ k & \text{otherwise}. \end{cases}$$

Equation (14) states that the value function at time t equals the maximum over the set of trading strategies on $(t, h\tau)$ of the expectation of the continuation value at the end of the current reporting period (i.e., at time $h\tau$). In turn, the continuation value is given by the function v in (15). Its form depends on two factors:

- whether or not a loss exceeding the reported VaR is recorded at the end of the current reporting period (that is, whether or not $K^- - K(h\tau)$ exceeds VaR), and
- whether or not the end of the current reporting period coincides with the end of the current backtesting period (that is, whether or not $h\tau \in \mathcal{T}$).

If no exception is recorded in the current reporting period, the institution enters the next period with capital $K(h\tau)$, the amount of capital at the end of the current reporting period. On the other end, if an exception is recorded in the current reporting period, the institution is subject to a pecuniary penalty of $\lambda(K^- - VaR - K(h\tau))$, so that, given limited liability, it enters the next period with capital $(K(h\tau) - \lambda(K^- - VaR - K(h\tau)))^+$. In addition, if the end of the current reporting period does not coincide with the end of the current backtesting period and an exception is recorded in the current reporting period, the institution enters the next reporting period with one more exception (i is incremented by one). If the end of the current reporting period coincides with the end of the current backtesting period, the reserve multiple going into the next period is set to the new value $k(i)$ or $k(i+1)$ and the number of exceptions in the new backtesting period is reset to zero. In all cases, the VaR reported at the beginning of the new period is determinad optimally so as to maximize the continuation value.

Equations (14) and (15) make it possible to compute the value function V recursively using the terminal condition at T:

$$\begin{aligned} & v(K, K^-, VaR, i, k, T) \\ & \quad = \quad u(K)1_{\{K \geq K^- - VaR\}} + u(K - \lambda(K^- - VaR - K))1_{\{K < K^- - VaR\}} \end{aligned}$$

and solving the continuous-time optimal investment problem in (14) backward one reporting period at a time. We therefore focus below on this problem, which we approach using the martingale duality technique of Cvitanić and Karatzas (1992) and Cuoco (1997).[22]

Remark 1 *The capital requirement constraint in (14) implies*

$$K(t) \geq k\,VaR \geq 0 \quad \text{for all } t \in [(h-1)\tau, h\tau). \tag{16}$$

Hence, the maximum possible loss $K^- - K(h\tau)$ over the reporting period cannot exceed $K^- - k\,VaR$. Clearly, reporting a VaR equal to the maximum possible loss over the reporting period is sufficient to avoid all the penalties, while reporting a VaR larger than this level is never optimal, since it provides no additional benefit, but it increases the capital reserve requirement. This implies

$$VaR \leq K^- - k\,VaR,$$

or

$$VaR \leq \frac{K^-}{1+k}. \tag{17}$$

Remark 2 *As a consequence of the inequality (16), capital must be non-negative at all times during the generic reporting period. However, default is still possible at the end of the reporting period if a violation is recorded and the institution's capital is insufficient to cover the associated pecuniary penalty, that is, if*

$$K(h\tau) \leq \lambda(K^- - VaR - K(h\tau)),$$

[22] Cvitanić and Karatzas (1992) develop the martingale duality technique for a class of investment problems involving convex constraints on the portfolio weights. The portfolio constraint in (14) involves the total value of capital K and thus is not included in the setup considered by Cvitanić and Karatzas (1992). Cuoco (1997) provides an extension of the martingale duality technique to a more general class of convex constraints on the portfolio amounts. This class of constraints includes the one in (14).

or

$$K(h\tau) \leq \frac{\lambda(K^- - VaR)}{1+\lambda}.$$

3.2 PDE Characterization of the Value Function

Let

$$\tilde{A} = \Big\{(\nu_0, \nu_-) \in \mathbb{R} \times \mathbb{R}^m : \nu_0 \geq 0,\ \nu_0(\bar{1} - \beta) \leq \nu_- \leq \nu_0(\bar{1} + \beta)\Big\},$$

and let $\mathcal{N}$ denote the set of $\tilde{A}$-valued bounded processes on $[0, \tau)$. Finally, for $\nu \in \mathcal{N}$ let

$$\xi_\nu(t) = \exp\left(-\int_0^t \left(r + \nu_0(s) + \frac{|\kappa_\nu(s)|^2}{2}\right) ds - \int_0^t \kappa_\nu(s)^\top dw(s)\right), \tag{18}$$

where

$$\kappa_\nu = \sigma^{-1}\left(\mu + \nu_- - \nu_0 \bar{1}\right),$$

The following result is then easily derived from Proposition 1 in Cuoco (1997).

Proposition 3 *Let*

$$\tilde{v}(z, K^-, VaR, i, k, h\tau) = \max_{K \geq k\,VaR}\left[v(K, K^-, VaR, i, k, h\tau) - zK\right] \tag{19}$$

and consider the problem

$$\min_{\nu \in \mathcal{N}} \mathrm{E}\Bigg[\tilde{v}(\psi\xi_\nu(h\tau), K^-, VaR, i, k, h\tau) - \psi\left(k\,VaR \int_{(h-1)\tau}^{h\tau} \xi_\nu(s)\nu_0(s)\,ds - K^-\right) \Bigg|\ Z_\nu((h-1)\tau) = \psi\Bigg].$$

If the above problem has a solution for all $\psi > 0$, then

$$V(K, K^-, VaR, i, k, t) = \min_{\psi > 0}\Big[\tilde{V}(\psi, K^-, VaR, i, k, t) + \psi K\Big] \tag{20}$$

for all $K \geq k\,VaR$ and all $t \in [(h-1)\tau, h\tau)$, where

$$\begin{aligned} &\tilde{V}(z, K^-, VaR, i, k, t) \\ &\quad = \min_{\nu \in \mathcal{N}} \mathrm{E}\Bigg[\tilde{v}(Z_\nu(\tau), K^-, VaR, i, k, h\tau) - k\,VaR \int_t^{h\tau} Z_\nu(s)\nu_0(s)\,ds\ \Bigg|\ Z_\nu(t) = z\Bigg] \\ &\quad \text{s.t. } dZ_\nu(t) = -Z_\nu(t)\Big((r + \nu_0(t))\,dt + \kappa_\nu(t)^\top dw(t)\Big). \end{aligned} \tag{21}$$

While in the unconstrained case it was possible to directly compute the dual value function $\tilde{V}^U$, an analytic solution is not available in the constrained case. However, the dual value function $\tilde{V}$ can be computed numerically by solving the associated Hamilton-Jacobi-Bellman (HJB) equation. Below we denote by ι_i the i-th column of the $m \times m$ identity matrix.

Proposition 4 *If $\beta \in I\!R^m_{++}$ the dual value function $\tilde{V}$ in (21) is strictly decreasing and strictly convex in z for all $t \in [(h-1)\tau, h\tau)$ and it solves the HJB equation*

$$0 = \tilde{V}_t - rz\tilde{V}_z + z^2\tilde{V}_{zz} \min_{\nu \in \tilde{A}} \left[\frac{1}{2}|\sigma^{-1}(\mu + \nu_- - \nu_0 \bar{1})|^2 - \frac{\tilde{V}_z + k\,VaR}{z\tilde{V}_{zz}} \nu_0 \right] \tag{22}$$

with terminal condition

$$\tilde{V}(z, K^-, VaR, i, k, h\tau) = \tilde{v}(z, K^-, VaR, i, k, h\tau).$$

Moreover, the process ν^* attaining the minimum in (22) satisfies

$$0 \leq \nu_0^* \leq M$$

and

$$M(\bar{1} - \beta) \leq \nu_- \leq M(\bar{1} + \beta),$$

where $M = \max\{|\iota_i^\top \mu| / \iota_i^\top \beta : i = 1, \ldots, m\}$. Hence, $\nu^* \in \mathcal{N}$.

Remark 3 *The terminal condition for $\tilde{V}$ in Proposition 4 is given by the function $\tilde{v}$ defined in (19). An explicit expression for $\tilde{v}(z, K^-, VaR, i, k, h\tau)$ in terms of the initial value $\tilde{V}(z, K^-, VaR, i, k, h\tau)$ of the dual value function computed over the next reporting period $[h\tau, (h+1)\tau)$ is provided in Appendix C. This expression simplifies the recursive solution of the dual value function.*

3.3 Optimal Investment Strategy

Once the dual value function $\tilde{V}$ is known, the optimal trading strategy θ and the process for the institution's capital K can be recovered as in Proposition 2. To prevent excessively cumbersome notation, we suppress from now on the dependence of the dual value function on the variables (K^-, VaR, i, k) which are constant within each reporting period.

Proposition 5 *If $\beta \in I\!R^m_{++}$, the optimal trading strategy θ for the constrained problem in (14) is given by $\theta(t) = \theta(Z_{\nu^*}(t), t)$, where*

$$\theta(z, t) = z\tilde{V}_{zz}(z, t)(\sigma\sigma^\top)^{-1}(\mu + \nu_-^*(z, t) - \nu_0^*(z, t)\bar{1}). \tag{23}$$

Moreover, under the optimal trading strategy, the institution's capital at time t is given by $K(t) = K(Z_{\nu^*}(t), t)$, where

$$K(z, t) = -\tilde{V}_z(z, t). \tag{24}$$

In particular, $K(z, h\tau) = f(z)$, where f is the function defined in Appendix C.

A comparison of equations (13) and (23) shows that the constrained optimal trading strategy coincides with the unconstrained optimal trading strategy in a fictitious economy in which the vector of stock risk premia equals $\mu + \nu_-^* - \nu_0^* \bar{1}$. The next proposition provides an explicit characterization of the optimal trading strategy in the presence of capital requirements. We denote by I_i the $i \times m$ matrix consisting of the first i rows of the $m \times m$ identity matrix.

Proposition 6 *Suppose that $\beta \in I\!R^m_{++}$ and that all the components of the unconstrained mean-variance efficient portfolio $(\sigma\sigma^\top)^{-1}\mu$ are different from zero.*[23] *For $i, j \in \{1, 2, \ldots, m\}$, $j \leq i$, let*

$$\eta_{i,j} = \frac{\iota_j^\top H I_i^\top (I_i \sigma\sigma^\top I_i^\top)^{-1} I_i \mu}{\iota_j^\top H I_i^\top (I_i \sigma\sigma^\top I_i^\top)^{-1} I_i H \beta},$$

where

$$H = \mathrm{diag}\left(\mathrm{sign}\left((\sigma\sigma^\top)^{-1}\mu\right)\right)$$

and suppose without loss of generality that the assets are sorted so that

$$\eta_{i,i} = \min\left\{\eta_{i,j} : \eta_{i,j} > 0,\ j = 1, \ldots, i\right\}.$$

For $i = 1, 2, \ldots, m$, let

$$h_i = \beta^\top H I_i^\top (I_i \sigma\sigma^\top I_i^\top)^{-1} I_i (\mu - \eta_{i,i} H \beta)$$

and let

$$h_{m+1} = \beta^\top H (\sigma\sigma^\top)^{-1}\mu.$$

Then

$$0 = h_1 \leq h_2 \leq \ldots \leq h_{m+1}.$$

If

$$-\frac{\tilde{V}_z(z,t) + k\,VaR}{z\tilde{V}_{zz}(z,t)} \geq h_{m+1},$$

then $\nu_0^*(z,t) = 0$ and

$$\theta(z,t) = z\tilde{V}_{zz}(z,t)(\sigma\sigma^\top)^{-1}\mu \tag{25}$$

If

$$h_i \leq -\frac{\tilde{V}_z(z,t) + k\,VaR}{z\tilde{V}_{zz}(z,t)} < h_{i+1},$$

for $i = 1, 2, \ldots, m$ then

$$\nu_0^*(z,t) = \frac{\beta^\top H I_i^\top (I_i \sigma\sigma^\top I_i^\top)^{-1} I_i \mu + \frac{\tilde{V}_z(z,t) + k\,VaR}{z\tilde{V}_{zz}(z,t)}}{\beta^\top H I_i^\top (I_i \sigma\sigma^\top I_i^\top)^{-1} I_i H \beta} \tag{26}$$

and

$$\theta(z,t) = z\tilde{V}_{zz}(z,t) I_i^\top (I_i \sigma\sigma^\top I_i^\top)^{-1} I_i \left(\mu - \nu_0^*(z,t) H \beta\right), \tag{27}$$

In particular, $H\theta \geq 0$, that is, the components of the constrained optimal portfolio never have the opposite sign of the corresponding components of the mean-variance efficient portfolio.

[23] If an asset is not included in the unconstrained mean-variance efficient portfolio, it would not be included in the constrained portfolio and thus can be ignored.

The fact that the components of the constrained optimal portfolio never have the opposite sign of the corresponding components of the mean-variance efficient portfolio implies that the nonlinear portfolio constraint in (14) is equivalent to the pair of linear constraints

$$H\theta(s) \geq 0 \tag{28}$$

$$K(s) \geq k\,VaR + \beta^\top H\theta(s). \tag{29}$$

The characterization of the optimal portfolio strategy in the previous Proposition is then quite intuitive: as long as the non-negativity constraint in (28) is not binding, constrained optimal portfolios are combinations of the portfolio that maximizes expected return for a given variance (the mean-variance efficient portfolio) and the portfolio that minimizes the charge for credit risk $\beta^\top H\theta$ for a given variance: we refer to the latter portfolio as the constrained minimum-variance portfolio, since it is also the portfolio that minimizes variance subject to a constraint on the charge for credit risk. More generally, for $h_i < -\frac{\tilde{V}_z(z,t)+k\,VaR}{z\tilde{V}_{zz}(z,t)} \leq h_{i+1}$ the non-negativity constraint in (28) binds for the last $m-i$ assets, so that $\iota_j^\top\theta(z,t) = 0$ for $j = m-i,\ldots,m$ and (as shown in equation (27))

$$\theta(z,t) = z\tilde{V}_{zz}(z,t)\pi_i^{\mathrm{MVE}} - z\tilde{V}_{zz}(z,t)\nu_0^*(z,t)\pi_i^{\mathrm{CMV}},$$

where

$$\pi_i^{\mathrm{MVE}} = (I_i\sigma\sigma^\top I_i^\top)^{-1}I_i\mu$$

denotes the mean-variance efficient portfolio of the first i risky assets and

$$\pi_i^{\mathrm{CMV}} = (I_i\sigma\sigma^\top I_i^\top)^{-1}I_iH\beta$$

denotes the constrained minimum-variance portfolio of the first i risky assets.

Both $h(z,t) = -\frac{\tilde{V}_z(z,t)+k\,VaR}{z\tilde{V}_{zz}(z,t)}$ and $K(z,t) = -\tilde{V}_z(z,t)$ are monotonically decreasing functions of z.[24] Thus, Proposition 6 shows that when capital $K(Z_{\nu^*}(t),t)$ is large (that is, when $Z_{\nu^*}(t)$ is small and $h(Z_{\nu^*}(t),t) \geq h_{m+1}$), the capital constraint does not bind ($\nu_0^*(Z_{\nu^*}(t),t) = 0$) and the financial institution holds the mean-variance efficient portfolio of risky assets π_m^{MVE}. For lower levels of capital (that is, when $h(Z_{\nu^*}(t),t) < h_{m+1}$), the constraint starts to bind ($\nu_0^*(Z_{\nu^*}(t),t)$ becomes positive) and the institution is forced to alter its leverage to satisfy the constraint. At the same time, it finds it optimal to rebalance its portfolio of risky assets: this rebalancing is done by shorting the constrained minimum-variance portfolio π_m^{CMV}.

For lower levels of capital (higher values of $Z_{\nu^*}(t)$), shorting of the corrective portfolio progressively increases, until the institution reaches a point where its investment in the m-th asset is zero (this happens when $h(Z_{\nu^*}(t),t) = h_m$). Beyond this point, the institution simply drops the m-th asset from its portfolio, since it is never optimal to short (respectively, long) an asset that is held in positive (respectively, negative) amounts in the unconstrained mean-variance efficient portfolio. If the assets are uncorrelated (σ is diagonal) the m-th asset is the one with the lowest ratio of absolute risk premium $|\iota_j^\top\mu|$ to risk weight $\iota_j^\top\beta$.

[24]The term ν_0^* can be interpreted as the Lagrangian multiplier on the constraint in the primal problem (14): thus ν_0^* is inversely related to capital K. Since h is a decreasing function of ν_0^* (as shown in equation (26)), h must be an increasing function of K (that is, a decreasing function of z).

Thus, as the institution is forced to reduce its allocation to risky assets to satisfy the capital constraint, it finds it optimal to tilt its portfolio toward assets with high absolute risk premia and low risk weights.[25] If the assets are correlated, then correlations are also taken into account in deciding which asset is dropped first from the portfolio, and the m-th asset is the one with the lowest ratio $\eta_{m,j}$.

If capital decreases ($Z_{\nu^*}(t)$ increases) even further and the constraint becomes even more severe, the institution sequentially drops other risky assets from its portfolio, concentrating on those characterized by progressively higher absolute risk premia and lower risk weights. This happens each time that $h(Z_{\nu^*}(t), t)$ exceeds a new value h_j. Eventually, if $h(Z_{\nu^*}(t), t) = h_1 = 0$ (that is, if $K(Z_{\nu^*}(t), t) = k\,VaR$), the institution is forced to invest its entire portfolio in the riskless asset. In general, whenever $h(Z_{\nu^*}(t), t)$ is between h_i and h_{i+1}, the institution only holds the first i risky assets and its portfolio is a combination of the riskless asset and two funds of risky assets: the mean-variance efficient portfolio of the first i assets, π_i^{MVE} and the constrained minimum-variance portfolio of the first i assets, π_i^{CMV}. Thus, locally (that is, between any pair h_i and h_{i+1}), optimal portfolios satisfy three-fund separation.

3.4 The One-Dimensional Case

Not surprisingly, the results in the previous two subsections take a very simple form in the case of a single risky asset, as shown in the following Corollary.

Corollary 2 *In the case of a single risky asset ($m = 1$) and a positive risk premium ($\mu > 0$) the HJB equation (22) reduces to*

$$\begin{aligned} 0 &= \tilde{V}_t - rz\tilde{V}_z + z^2\tilde{V}_{zz}\,\min\left[\left(\frac{\mu}{\sigma}\right)^2,\, -\frac{\mu}{\beta}\left(\frac{\tilde{V}_z + k\,VaR}{z\tilde{V}_{zz}(z,t)}\right)\right] \\ &\quad - \frac{1}{2}z^2\tilde{V}_{zz}\,\min\left[\left(\frac{\mu}{\sigma}\right)^2,\, \left(\frac{\sigma}{\beta}\right)^2\left(\frac{\tilde{V}_z + k\,VaR}{z\tilde{V}_{zz}(z,t)}\right)^2\right]. \end{aligned}$$

Moreover, the optimal investment strategy in (23) reduces to

$$\begin{aligned} \theta(z,t) &= \min\left[z\tilde{V}_{zz}(z,t)\frac{\mu}{\sigma^2},\, \frac{1}{\beta}\left(K(z,t) - k\,VaR\right)\right] \\ &= \min\left[\frac{1}{\Gamma(z,t)}\frac{\mu}{\sigma^2}K(z,t),\, \frac{1}{\beta}\left(K(z,t) - k\,VaR\right)\right], \end{aligned} \tag{30}$$

where

$$\Gamma(z,t) = -\frac{\tilde{V}_z(z,t)}{z\tilde{V}_{zz}(z,t)}$$

is the primal value function relative risk aversion coefficient.[26]

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[25] This substitution effect is similar to the one described by Kohen and Santomero (1980), Kim and Santomero (1988) and Rochet (1992) in a static setting and by Blum (1999) in a two-period setting.

[26] By duality,

$$z = V_K(K(z,t),t) = V_K(-\tilde{V}_z(z,t),t)$$

4 Analysis of Optimal Policies

For our numerical analysis, we fix throughout the backtesting period to one year, the investment horizon to two years ($T = 2$) and the reporting period to one day ($n = 250$, $\tau = 1/250$) and we assume that the reserve multiplier over the second year is determined according to the schedule proposed by the Basel Committee multiplied by the square root of ten,[27] that is,

$$k(i) = \begin{cases} 3.00\sqrt{10} = \ \ 9.49 & \text{if } i \leq \ 4 \\ 3.40\sqrt{10} = 10.75 & \text{if } i = \ 5 \\ 3.50\sqrt{10} = 11.07 & \text{if } i = \ 6 \\ 3.65\sqrt{10} = 11.54 & \text{if } i = \ 7 \\ 3.75\sqrt{10} = 11.86 & \text{if } i = \ 8 \\ 3.85\sqrt{10} = 12.17 & \text{if } i = \ 9 \\ 4.00\sqrt{10} = 12.65 & \text{if } i \geq 10. \end{cases}$$

We consider the cases in which the number m of risky assets equals 1 or 2.[28] In either case, we set $r = 0$ and choose the vector of risk premia μ and the volatility matrix σ so that the risk premium (respectively, the volatility) of the mean-variance efficient portfolio of risky assets equals 0.059 (respectively, 0.22).[29] In the case of two risky assets, we assume in addition that the volatility of the first asset (respectively, the second asset) is 25% higher (respectively, 25% lower) than the volatility of the mean-variance efficient portfolio and that the correlation coefficients between the returns on the two assets is 0.50.[30]

We solve for the dual value function recursively as explained in the previous section by numerically integrating the PDE (22) using the method of lines.[31] We also compute the distribution function P of the state variable Z_{ν^*} at the end of each reporting period by using the method of lines to solve the PDE

$$\begin{gathered} \frac{\partial}{\partial t} P(z,t) = \frac{1}{2}\frac{\partial}{\partial z}\Big(|\kappa_{\nu^*}(z,t)z|^2 P_z(z,t)\Big) + \Big(r + \nu_0^*(z,t)\Big) z P_z(z,t), \\ P(z,(h-1)\tau) = 1_{\{z \geq \psi^*\}}, \end{gathered} \tag{31}$$

---

and hence (differentiating with respect to z and rearranging)

$$-\frac{\tilde{V}_z(z,t)}{z\tilde{V}_{zz}(z,t)} = -\frac{K(z,t)V_{KK}(K(z,t),t)}{V_K(K(z,t),t)}.$$

[27] See footnote 8.

[28] As noted at the end of the previous section, optimal investment policies satisfy local three-fund separation, so that considering additional risky assets would not affect the analysis.

[29] These values correspond to the mean risk premium and the return standard deviation of the market portfolio as estimated by Ibbotson and Sinquefeld (1982).

[30] These assumptions imply that $\mu = \begin{pmatrix} .07225 \\ .02937 \end{pmatrix}$ and $\sigma = \begin{pmatrix} .27500 & .00000 \\ .08250 & .14289 \end{pmatrix}$ in our simulations with $m = 2$.

[31] The method of lines discretizes the spatial variable z and replaces partial derivatives with finite differences to generate a system of first-order ODE's in the time variable that can be solved backward starting from the terminal condition.

for $t \in [(h-1)\tau, h\tau)$, where ψ^* is the value of ψ solving (20) with $t = (h-1)\tau$.[32] This allows us to compute the distribution of the financial institution's capital using equation (24), and hence the true VaR of the portfolio, which we compare to the reported VaR. For comparison purposes, we also compute the optimal policies and the true VaR for the unconstrained problem described in Section 2.

4.1 One Risky Asset

Table 1 shows the optimal reporting and investment strategy at the beginning of the first and of the last reporting period in the first year ($t = 0$ and $t = 249/250 = .996$, respectively) for three different values of the number violations in the current backtesting period ($i = 0$, $i = 5$ and $i = 9$), two different values of the current reserve multiplier ($k = 9.49$ and $k = 12.65$) and three different values of the current leverage ratio $\text{LR} = D/(D+K)$,[33] for the case $\gamma = .25$, $\varepsilon = 0$, $\beta = .08$,[34] $\lambda = .01$ and $m = 1$. For each combination of (t, i, k, LR), the table shows the reported VaR normalized by the total asset value $v = VaR/(D+K)$, the coefficient of relative risk aversion of the primal value function $\Gamma = -KV_{KK}/V_K$, the maximum possible proportional allocation to risky assets under the capital requirent constraint $\bar{\pi} = (1 - \text{LR} - kv)/\beta$,[35] the proportional allocation to risky assets $\pi = \theta/(D+K)$ and the true 1-day 90% and 99% VaRs normalized by the total asset value $v_{.90}$ and $v_{.99}$. For comparison purposes, the table also shows the relative risk aversion coefficient Γ^U, the proportional allocation to risky assets π^U and the true normalized 1-day 90% and 99% VaRs $v^U_{.90}$ and $v^U_{.99}$ in the unconstrained case. Table 2 shows the same information for a higher value of the institution's risk aversion coefficient ($\gamma = .50$).

For the set of parameters considered in Table 1, the proportional allocation to risky assets in the absence of capital requirements varies between 482.7% and 820.2%, increasing when the leverage ratio increases (asset value decreases) and the option to default becomes more valuable (as shown in Figure 1). Capital requirements are effective in curbing the risk of trading portfolios, reducing the range of the proportional allocation to risky assets to between 18.2% and 412.2%. Not surprisingly, this risk reduction is larger when capital requirements are more stringent (that is, when the reserve multiplier k is larger). Moreover, the allocation to risky assets becomes inversely related to leverage ratios, since as capital falls and leverage ratios increase, financial institutions are forced to liquidate their holdings of risky assets to avoid violating capital requirements. As a result of this dynamic behavior, the range of true 99% daily VaRs falls from between 14.60% and 24.15% in the unconstrained

[32] The PDE in (31) is obtained by integrating the forward Kolmogorov equation

$$\frac{\partial}{\partial t} p(z,t) = \frac{1}{2}\frac{\partial^2}{\partial z^2}\left(|\kappa_{\nu^*}(z,t)z|^2 p(z,t)\right) + \frac{\partial}{\partial z}\left((r + \nu_0^*(z,t))zp(z,t)\right)$$

solved by the density function $p(z,t) = P_z(z,t)$.

[33] The exact values of the three leverage ratios used for all the tables in the paper are $1/(1+e^3) = .0474$, $1/(1+e^0) = .5000$ and $1/(1+e^{-3}) = .9526$ before rounding.

[34] See footnote 7.

[35] The constraint in (14) implies

$$\frac{\theta}{D+K} \le \frac{K - k\,VaR}{\beta(D+K)} = \frac{1 - \text{LR} - kv}{\beta}.$$

Parameter values: $\gamma = .25$, $\varepsilon = 0$, $\beta = .08$, $\lambda = .01$, $\tau = .004$, $m = 1$

t	i	k	LR	v	Γ	$\bar{\pi}$	π	$v_{.90}$	$v_{.99}$	Γ^U	π^U	$v^U_{.90}$	$v^U_{.99}$
0	0	9.49	.05	.0663	.2862	4.243	4.041	.0663	.0663	.2357	4.927	.0853	.1488
0	0	9.49	.50	.0348	.2862	4.243	2.121	.0348	.0348	.0991	6.148	.1062	.1842
0	0	9.49	.95	.0033	.2862	4.243	0.201	.0033	.0033	.0086	6.698	.1155	.1994
0	0	12.65	.05	.0515	.3133	3.952	3.707	.0515	.0515	.2357	4.927	.0853	.1488
0	0	12.65	.50	.0270	.3133	3.952	1.945	.0270	.0270	.0991	6.148	.1062	.1842
0	0	12.65	.95	.0026	.3133	3.952	0.185	.0026	.0026	.0086	6.698	.1155	.1994
0	5	9.49	.05	.0667	.2920	4.197	3.976	.0667	.0667	.2357	4.927	.0853	.1488
0	5	9.49	.50	.0350	.2921	4.197	2.087	.0350	.0350	.0991	6.148	.1062	.1842
0	5	9.49	.95	.0033	.2921	4.197	0.198	.0033	.0033	.0086	6.698	.1155	.1994
0	5	12.65	.05	.0505	.2914	4.126	3.930	.0505	.0977	.2357	4.927	.0853	.1488
0	5	12.65	.50	.0265	.2914	4.126	2.063	.0265	.0516	.0991	6.148	.1062	.1842
0	5	12.65	.95	.0025	.2914	4.126	0.196	.0025	.0049	.0086	6.698	.1155	.1994
0	10	9.49	.05	.0656	.2781	4.327	4.122	.0656	.1012	.2357	4.927	.0853	.1488
0	10	9.49	.50	.0345	.2781	4.327	2.164	.0345	.0534	.0991	6.148	.1062	.1842
0	10	9.49	.95	.0033	.2781	4.327	0.205	.0033	.0050	.0086	6.698	.1155	.1994
0	10	12.65	.05	.0504	.2913	4.127	3.931	.0504	.0976	.2357	4.927	.0853	.1488
0	10	12.65	.50	.0265	.2914	4.127	2.063	.0265	.0513	.0991	6.148	.1062	.1842
0	10	12.65	.95	.0025	.2914	4.127	0.196	.0025	.0049	.0086	6.698	.1155	.1994
.996	0	9.49	.05	.0656	.2781	4.327	4.122	.0656	.1016	.2405	4.827	.0836	.1460
.996	0	9.49	.50	.0345	.2781	4.327	2.164	.0345	.0533	.0850	7.169	.1237	.2139
.996	0	9.49	.95	.0033	.2781	4.327	0.205	.0033	.0050	.0070	8.202	.1410	.2415
.996	0	12.65	.05	.0504	.2913	4.127	3.931	.0504	.0860	.2405	4.827	.0836	.1460
.996	0	12.65	.50	.0265	.2914	4.127	2.063	.0265	.0452	.0850	7.169	.1237	.2139
.996	0	12.65	.95	.0025	.2914	4.127	0.196	.0025	.0043	.0070	8.202	.1410	.2415
.996	5	9.49	.05	.0667	.2915	4.202	3.984	.0667	.0667	.2405	4.827	.0836	.1460
.996	5	9.49	.50	.0350	.2915	4.202	2.091	.0350	.0350	.0850	7.169	.1237	.2139
.996	5	9.49	.95	.0033	.2915	4.202	0.198	.0033	.0033	.0070	8.202	.1410	.2415
.996	5	12.65	.05	.0517	.3171	3.923	3.662	.0517	.0517	.2405	4.827	.0836	.1460
.996	5	12.65	.50	.0271	.3172	3.923	1.922	.0271	.0271	.0850	7.169	.1237	.2139
.996	5	12.65	.95	.0026	.3172	3.923	0.182	.0026	.0026	.0070	8.202	.1410	.2415
.996	10	9.49	.05	.0656	.2781	4.327	4.122	.0656	.1129	.2405	4.827	.0836	.1460
.996	10	9.49	.50	.0345	.2781	4.327	2.164	.0345	.0592	.0850	7.169	.1237	.2139
.996	10	9.49	.95	.0033	.2781	4.327	0.205	.0033	.0056	.0070	8.202	.1410	.2415
.996	10	12.65	.05	.0504	.2913	4.127	3.931	.0504	.0976	.2405	4.827	.0836	.1460
.996	10	12.65	.50	.0265	.2913	4.127	2.063	.0265	.0512	.0850	7.169	.1237	.2139
.996	10	12.65	.95	.0025	.2914	4.127	0.196	.0025	.0049	.0070	8.202	.1410	.2415

Table 1: The table shows the values of the reported normalized daily VaR (v), the RRA coefficient of the dual value function (Γ), the upper bound on the portfolio allocation to the risky asset ($\bar{\pi}$), the portfolio allocation to the risky asset (π), the true normalized 90% VaR ($v_{.90}$) and the true normalized 99% VaR ($v_{.99}$) in the presence of capital requirements, as well as the RRA coefficient (Γ^U), the portfolio allocation to the risky asset (π^U) and the true normalized 90% and 99% VaRs ($v^U_{.90}$ and $v^U_{.99}$) in the unconstrained case.

Parameter values: $\gamma = .50$, $\varepsilon = 0$, $\beta = .08$, $\lambda = .01$, $\tau = .004$, $m = 1$													
t	i	k	LR	v	Γ	$\bar{\pi}$	π	$v_{.90}$	$v_{.99}$	Γ^U	π^U	$v^U_{.90}$	$v^U_{.99}$
0	0	9.49	.05	.0777	.5025	2.830	2.311	.0403	.0706	.5000	2.323	.0405	.0723
0	0	9.49	.50	.0408	.5024	2.830	1.213	.0213	.0370	.2078	2.933	.0513	.0923
0	0	9.49	.95	.0039	.5024	2.830	0.115	.0020	.0035	.0149	3.892	.0679	.1212
0	0	12.65	.05	.0596	.5113	2.601	2.271	.0396	.0596	.5000	2.323	.0405	.0723
0	0	12.65	.50	.0313	.5113	2.601	1.192	.0209	.0313	.2078	2.933	.0513	.0923
0	0	12.65	.95	.0030	.5113	2.601	0.113	.0020	.0030	.0149	3.892	.0679	.1212
0	5	9.49	.05	.0777	.5027	2.826	2.310	.0404	.0707	.5000	2.323	.0405	.0723
0	5	9.49	.50	.0408	.5026	2.826	1.213	.0213	.0370	.2078	2.933	.0513	.0923
0	5	9.49	.95	.0039	.5026	2.826	0.115	.0020	.0035	.0149	3.892	.0679	.1212
0	5	12.65	.05	.0597	.5125	2.595	2.266	.0396	.0597	.5000	2.323	.0405	.0723
0	5	12.65	.50	.0313	.5124	2.595	1.189	.0209	.0313	.2078	2.933	.0513	.0923
0	5	12.65	.95	.0030	.5124	2.595	0.113	.0020	.0030	.0149	3.892	.0679	.1212
0	10	9.49	.05	.0777	.5025	2.830	2.311	.0404	.0706	.5000	2.323	.0405	.0723
0	10	9.49	.50	.0408	.5024	2.830	1.213	.0213	.0370	.2078	2.933	.0513	.0923
0	10	9.49	.95	.0039	.5024	2.830	0.115	.0020	.0035	.0149	3.892	.0679	.1212
0	10	12.65	.05	.0596	.5113	2.601	2.271	.0396	.0596	.5000	2.323	.0405	.0723
0	10	12.65	.50	.0313	.5112	2.601	1.192	.0209	.0313	.2078	2.933	.0513	.0923
0	10	12.65	.95	.0030	.5112	2.601	0.113	.0020	.0030	.0149	3.892	.0679	.1212
.996	0	9.49	.05	.0777	.5025	2.830	2.311	.0405	.0706	.5000	2.322	.0405	.0723
.996	0	9.49	.50	.0408	.5024	2.830	1.213	.0213	.0372	.1922	3.172	.0557	.1010
.996	0	9.49	.95	.0039	.5024	2.830	0.115	.0020	.0035	.0119	4.845	.0846	.1509
.996	0	12.65	.05	.0596	.5113	2.601	2.271	.0393	.0596	.5000	2.322	.0405	.0723
.996	0	12.65	.50	.0313	.5113	2.601	1.192	.0207	.0313	.1922	3.172	.0557	.1010
.996	0	12.65	.95	.0030	.5113	2.601	0.113	.0020	.0030	.0119	4.845	.0846	.1509
.996	5	9.49	.05	.0777	.5027	2.826	2.310	.0407	.0708	.5000	2.322	.0405	.0723
.996	5	9.49	.50	.0408	.5026	2.826	1.213	.0213	.0372	.1922	3.172	.0557	.1010
.996	5	9.49	.95	.0039	.5026	2.826	0.115	.0020	.0035	.0119	4.845	.0846	.1509
.996	5	12.65	.05	.0597	.5123	2.595	2.267	.0394	.0597	.5000	2.322	.0405	.0723
.996	5	12.65	.50	.0313	.5123	2.595	1.190	.0208	.0313	.1922	3.172	.0557	.1010
.996	5	12.65	.95	.0030	.5123	2.595	0.113	.0020	.0030	.0119	4.845	.0846	.1509
.996	10	9.49	.05	.0777	.5025	2.830	2.311	.0407	.0709	.5000	2.322	.0405	.0723
.996	10	9.49	.50	.0408	.5024	2.830	1.213	.0215	.0374	.1922	3.172	.0557	.1010
.996	10	9.49	.95	.0039	.5024	2.830	0.115	.0020	.0035	.0119	4.845	.0846	.1509
.996	10	12.65	.05	.0596	.5113	2.601	2.271	.0398	.0596	.5000	2.322	.0405	.0723
.996	10	12.65	.50	.0313	.5113	2.601	1.192	.0209	.0313	.1922	3.172	.0557	.1010
.996	10	12.65	.95	.0030	.5113	2.601	0.113	.0020	.0030	.0119	4.845	.0846	.1509

Table 2: The table shows the values of the reported normalized daily VaR (v), the RRA coefficient of the dual value function (Γ), the upper bound on the portfolio allocation to the risky asset ($\bar{\pi}$), the portfolio allocation to the risky asset (π), the true normalized 90% VaR ($v_{.90}$) and the true normalized 99% VaR ($v_{.99}$) in the presence of capital requirements, as well as the RRA coefficient (Γ^U), the portfolio allocation to the risky asset (π^U) and the true normalized 90% and 99% VaRs ($v^U_{.90}$ and $v^U_{.99}$) in the unconstrained case.

case to between 0.26% and 11.29%. More importantly, while the probability of default at time T conditional on leverage at time 0 ranges from 12.33% when (LR=.05) to 67.41% (when LR=.95) in the absence of capital requirements, this probability is 0 in the presence of capital requirements.[36]

Reported VaRs range between 0.25% and 6.67%, showing a significant positive correlation with true 99% VaRs. The incentives to underreport true VaRs are stronger when k is large and capital requirements are more stringent or when the number of exceptions in the current backtesting period has already reached 10, so that additional exceptions imply no further increases in the reserve multiplier. As a result, for the set of parameters considered in Table 1, the financial institution is truthfully reporting its 99% VaR at $t = 0$ when $k = 9.49$ and i is less than 10 and under-reporting when $k = 12.65$ or $i = 10$ (essentially reporting a 90% VaR).

The incentives to under-report also depend critically on the institution's risk aversion. Since the welfare cost of capital requirements is larger the lower the risk aversion, while the welfare cost of the pecuniary penalty associated with an exception is lower the lower the risk aversion, institutions with low risk aversion have an incentive to under-report their true VaR and institutions with high risk aversion have an incentive to over-report. As shown in Table 2, an institution with a risk aversion coefficient of 0.50 (or larger) would never under-report its true VaR at $t = 0$ and possibly over-report.[37] As a result, reported VaR's display smaller variation across different risk aversions than true VaR's.

Reported VaRs also display an intuitive trend in the course of the backtesting period if the number i of recorded exceptions is kept constant. Since penalties in the form of higher reserve multipliers k are not imposed until the number i of exceptions exceeds 4, an institution with less than 4 recorded exceptions in the course of the current backtesting period will tend to increase its proportional allocation to risky assets and to decrease its reported VaRs over time. This is evident in Table 1 when comparing the investment and reporting strategies when $i = 0$ at the beginning of the backtesting period ($t = 0$) with those at the end of the backtesting period ($t = .996$). On the other hand, when $i = 10$, so that the institution is certain to be subject to a reserve multiplier equal to 12.65 in the next period, no significant time trend emerges. Finally, for a number of recorded exceptions between 5 and 9, the institution has incentives to reduce its true 99% VaRs and to increase its reported VaRs in the course of the backtesting period to avoid recording one additional exception that would translate into a larger value of the reserve multiplier k over the next backtesting period. Thus, in Table 1 when $i = 5$ the institution goes from under-reporting its VaR at the beginning of the backtesting period to truthfully reporting at the end of the backtesting period. As shown in Table 2, time variations in the course of the backtesting period also tend to become less significant as risk aversion increases.

[36] As noted in Remark 2, default is in principle still possible in the model with capital requirements if a loss exceeding the reported VaR is recorded at the end of a reporting period and capital is not sufficient to cover the pecuniary penalty associated with an exception. However, for all the parameters we consider, the institution optimally chooses to entirely avoid default in the presence of capital requirements.

[37] Berkowitz and O'Brien (2002) compare daily VaR reports with historical daily profit and losses for a sample of 6 large U.S. commercial banks and conclude that the banks in their sample tend to overestimate (or to over-report) their true VaR. Our results suggest that it should be possible to relate the extent of under- or over-reporting in a cross-section of financial institutions to the volatility of the institution's asset value (which proxies for the institution's risk aversion).

Parameter values: $\gamma = .25$, $\varepsilon = 0$, $\beta = .04$, $\lambda = .01$, $\tau = .004$, $m = 1$

t	i	k	LR	v	Γ	$\bar{\pi}$	π	$v_{.90}$	$v_{.99}$	Γ^U	π^U	$v^U_{.90}$	$v^U_{.99}$
0	0	9.49	.05	.0804	.2724	4.978	4.262	.0744	.0804	.2357	4.927	.0853	.1488
0	0	9.49	.50	.0422	.2724	4.978	2.237	.0395	.0422	.0991	6.148	.1062	.1842
0	0	9.49	.95	.0040	.2725	4.978	0.212	.0037	.0040	.0086	6.698	.1155	.1994
0	0	12.65	.05	.0618	.2950	4.495	3.936	.0618	.0618	.2357	4.927	.0853	.1488
0	0	12.65	.50	.0324	.2950	4.495	2.066	.0324	.0324	.0991	6.148	.1062	.1842
0	0	12.65	.95	.0031	.2951	4.495	0.196	.0031	.0031	.0086	6.698	.1155	.1994
0	5	9.49	.05	.0807	.2765	4.901	4.200	.0734	.0807	.2357	4.927	.0853	.1488
0	5	9.49	.50	.0424	.2765	4.901	2.204	.0388	.0424	.0991	6.148	.1062	.1842
0	5	9.49	.95	.0040	.2765	4.901	0.209	.0037	.0040	.0086	6.698	.1155	.1994
0	5	12.65	.05	.0620	.2973	4.414	3.906	.0620	.0620	.2357	4.927	.0853	.1488
0	5	12.65	.50	.0325	.2973	4.414	2.050	.0325	.0325	.0991	6.148	.1062	.1842
0	5	12.65	.95	.0031	.2974	4.414	0.194	.0031	.0031	.0086	6.698	.1155	.1994
0	10	9.49	.05	.0802	.2695	5.024	4.308	.0756	.1028	.2357	4.927	.0853	.1488
0	10	9.49	.50	.0421	.2695	5.024	2.261	.0394	.0538	.0991	6.148	.1062	.1842
0	10	9.49	.95	.0040	.2696	5.024	0.214	.0037	.0051	.0086	6.698	.1155	.1994
0	10	12.65	.05	.0614	.2837	4.613	4.093	.0614	.0955	.2357	4.927	.0853	.1488
0	10	12.65	.50	.0322	.2837	4.613	2.148	.0322	.0500	.0991	6.148	.1062	.1842
0	10	12.65	.95	.0031	.2838	4.613	0.204	.0031	.0047	.0086	6.698	.1155	.1994
.996	0	9.49	.05	.0802	.2695	5.024	4.308	.0757	.1027	.2357	4.827	.0836	.1460
.996	0	9.49	.50	.0421	.2695	5.024	2.261	.0395	.0538	.0850	7.169	.1237	.2139
.996	0	9.49	.95	.0040	.2695	5.024	0.214	.0037	.0051	.0070	8.202	.1410	.2415
.996	0	12.65	.05	.0614	.2837	4.613	4.093	.0614	.0882	.2357	4.827	.0836	.1460
.996	0	12.65	.50	.0322	.2837	4.613	2.148	.0322	.0464	.0850	7.169	.1237	.2139
.996	0	12.65	.95	.0031	.2838	4.613	0.204	.0031	.0044	.0070	8.202	.1410	.2415
.996	5	9.49	.05	.0807	.2762	4.904	4.204	.0774	.0807	.2357	4.827	.0836	.1460
.996	5	9.49	.50	.0424	.2763	4.904	2.206	.0406	.0424	.0850	7.169	.1237	.2139
.996	5	9.49	.95	.0040	.2763	4.904	0.209	.0040	.0040	.0070	8.202	.1410	.2415
.996	5	12.65	.05	.0621	.2991	4.389	3.882	.0618	.0621	.2357	4.827	.0836	.1460
.996	5	12.65	.50	.0326	.2991	4.389	2.038	.0326	.0326	.0850	7.169	.1237	.2139
.996	5	12.65	.95	.0031	.2992	4.389	0.193	.0031	.0031	.0070	8.202	.1410	.2415
.996	10	9.49	.05	.0802	.2695	5.024	4.308	.0802	.1103	.2357	4.827	.0836	.1460
.996	10	9.49	.50	.0421	.2695	5.024	2.261	.0421	.0578	.0850	7.169	.1237	.2139
.996	10	9.49	.95	.0040	.2695	5.024	0.214	.0040	.0055	.0070	8.202	.1410	.2415
.996	10	12.65	.05	.0614	.2837	4.613	4.093	.0614	.0954	.2357	4.827	.0836	.1460
.996	10	12.65	.50	.0322	.2837	4.613	2.148	.0322	.0501	.0850	7.169	.1237	.2139
.996	10	12.65	.95	.0031	.2838	4.613	0.204	.0031	.0047	.0070	8.202	.1410	.2415

Table 3: The table shows the values of the reported normalized daily VaR (v), the RRA coefficient of the dual value function (Γ), the upper bound on the portfolio allocation to the risky asset ($\bar{\pi}$), the portfolio allocation to the risky asset (π), the true normalized 90% VaR ($v_{.90}$) and the true normalized 99% VaR ($v_{.99}$) in the presence of capital requirements, as well as the RRA coefficient (Γ^U), the portfolio allocation to the risky asset (π^U) and the true normalized 90% and 99% VaRs ($v^U_{.90}$ and $v^U_{.99}$) in the unconstrained case.

Parameter values: $\gamma = .25$, $\varepsilon = 0$, $\beta = .08$, $\lambda = .05$, $\tau = .004$, $m = 1$

t	i	k	LR	v	Γ	$\bar{\pi}$	π	$v_{.90}$	$v_{.99}$	Γ^U	π^U	$v^U_{.90}$	$v^U_{.99}$
0	0	9.49	.05	.0669	.2958	4.171	3.926	.0669	.0669	.2357	4.927	.0853	.1488
0	0	9.49	.50	.0351	.2958	4.171	2.061	.0351	.0351	.0991	6.148	.1062	.1842
0	0	9.49	.95	.0033	.2958	4.171	0.195	.0033	.0033	.0086	6.698	.1155	.1994
0	0	12.65	.05	.0521	.3290	3.856	3.530	.0521	.0521	.2357	4.927	.0853	.1488
0	0	12.65	.50	.0273	.3290	3.856	1.853	.0273	.0273	.0991	6.148	.1062	.1842
0	0	12.65	.95	.0026	.3290	3.856	0.176	.0026	.0026	.0086	6.698	.1155	.1994
0	5	9.49	.05	.0669	.2958	4.171	3.926	.0669	.0669	.2357	4.927	.0853	.1488
0	5	9.49	.50	.0351	.2958	4.171	2.061	.0351	.0351	.0991	6.148	.1062	.1842
0	5	9.49	.95	.0033	.2958	4.171	0.195	.0033	.0033	.0086	6.698	.1155	.1994
0	5	12.65	.05	.0521	.3290	3.856	3.529	.0521	.0521	.2357	4.927	.0853	.1488
0	5	12.65	.50	.0273	.3290	3.856	1.852	.0273	.0273	.0991	6.148	.1062	.1842
0	5	12.65	.95	.0026	.3290	3.856	0.176	.0026	.0026	.0086	6.698	.1155	.1994
0	10	9.49	.05	.0669	.2958	4.171	3.926	.0669	.0669	.2357	4.927	.0853	.1488
0	10	9.49	.50	.0351	.2958	4.171	2.061	.0351	.0351	.0991	6.148	.1062	.1842
0	10	9.49	.95	.0033	.2958	4.171	0.195	.0033	.0033	.0086	6.698	.1155	.1994
0	10	12.65	.05	.0521	.3290	3.856	3.530	.0521	.0521	.2357	4.927	.0853	.1488
0	10	12.65	.50	.0273	.3290	3.856	1.853	.0273	.0273	.0991	6.148	.1062	.1842
0	10	12.65	.95	.0026	.3290	3.856	0.176	.0026	.0026	.0086	6.698	.1155	.1994
.996	0	9.49	.05	.0669	.2958	4.171	3.926	.0669	.0669	.2405	4.827	.0836	.1460
.996	0	9.49	.50	.0351	.2958	4.171	2.061	.0351	.0351	.0850	7.169	.1237	.2139
.996	0	9.49	.95	.0033	.2958	4.171	0.195	.0033	.0033	.0070	8.202	.1410	.2415
.996	0	12.65	.05	.0521	.3289	3.856	3.530	.0462	.0521	.2405	4.827	.0836	.1460
.996	0	12.65	.50	.0273	.3290	3.856	1.853	.0244	.0273	.0850	7.169	.1237	.2139
.996	0	12.65	.95	.0026	.3290	3.856	0.176	.0023	.0026	.0070	8.202	.1410	.2415
.996	5	9.49	.05	.0669	.2958	4.171	3.926	.0669	.0669	.2405	4.827	.0836	.1460
.996	5	9.49	.50	.0351	.2958	4.171	2.061	.0351	.0351	.0850	7.169	.1237	.2139
.996	5	9.49	.95	.0033	.2958	4.171	0.195	.0033	.0033	.0070	8.202	.1410	.2415
.996	5	12.65	.05	.0521	.3290	3.856	3.529	.0521	.0521	.2405	4.827	.0836	.1460
.996	5	12.65	.50	.0273	.3290	3.856	1.852	.0273	.0273	.0850	7.169	.1237	.2139
.996	5	12.65	.95	.0026	.3290	3.856	0.176	.0026	.0026	.0070	8.202	.1410	.2415
.996	10	9.49	.05	.0669	.2958	4.171	3.926	.0669	.0669	.2405	4.827	.0836	.1460
.996	10	9.49	.50	.0351	.2958	4.171	2.061	.0351	.0351	.0850	7.169	.1237	.2139
.996	10	9.49	.95	.0033	.2958	4.171	0.195	.0033	.0033	.0070	8.202	.1410	.2415
.996	10	12.65	.05	.0521	.3289	3.856	3.530	.0521	.0521	.2405	4.827	.0836	.1460
.996	10	12.65	.50	.0273	.3290	3.856	1.853	.0273	.0273	.0850	7.169	.1237	.2139
.996	10	12.65	.95	.0026	.3290	3.856	0.176	.0026	.0026	.0070	8.202	.1410	.2415

Table 4: The table shows the values of the reported normalized daily VaR (v), the RRA coefficient of the dual value function (Γ), the upper bound on the portfolio allocation to the risky asset ($\bar{\pi}$), the portfolio allocation to the risky asset (π), the true normalized 90% VaR ($v_{.90}$) and the true normalized 99% VaR ($v_{.99}$) in the presence of capital requirements, as well as the RRA coefficient (Γ^U), the portfolio allocation to the risky asset (π^U) and the true normalized 90% and 99% VaRs ($v^U_{.90}$ and $v^U_{.99}$) in the unconstrained case.

Overall, for the set of parameters considered in Tables 1 and 2, reported VaRs are never below true 90% VaRs.

Tables 3 and 4 show the impact on the optimal investment and reporting strategy of variations in the risk weight β and in the pecuniary penalty coefficient λ. A lower risk weight translates into less stringent capital requirements and hence into a lower opportunity cost of reporting high VaRs. The effect on risk-taking is ambiguous, since, as noted in the literature mentioned in the Introduction, lower capital requirements can reduce the marginal utility of capital in future periods and hence reduce the incentives to increase expected return by choosing highly leveraged portfolios. A comparison of Tables 1 and 3 confirms that while reported VaRs uniformly increase when the risk weight is reduced from 0.08 to 0.04, the impact on optimal risk-taking is ambiguous. Under-reporting is reduced (although not completely eliminated) with the lower risk weight.

On the other hand, as shown in Table 4, higher pecuniary penalties associated with exceptions lead to a decrease in risk-taking (as measured by the initial portfolio allocation π) and an increase in reported VaRs. Since the maximum possible period loss in the presence of capital requirements equals $K^- - k\,VaR$, the increase in reported VaRs implies a decrease in the true 100% VaRs, and hence a decrease in the extreme tail of the end-of-period asset value distribution. However, as shown in the table, the impact on VaRs at the 90% or even the 99% confidence level is ambiguous. In spite of this ambiguity, the degree of underreporting (as measured by the difference between true 99% VaRs and reported VaRs) uniformly decreases, with the institution now always truthfully reporting its 99% VaR.

Figures 2 and 3 plot the optimal portfolio weight π (heavier solid line) as a function of total asset value for the case $D = 1$, $K^- = 1$, $\beta = .08$, $\lambda = .01$, $\gamma = .25$, $\varepsilon = 0$, $i = 0$ and $k = 9.49$. Figure 2 plots the portfolio weight at a time close to the beginning of the first reporting period ($t = .001$), while Figure 3 plots the portfolio weight at a time close to the end of the first reporting period ($t = .003$)).

Equation (30) implies that, with a single risky asset, the portfolio weight equals the minimum of

$$\frac{1}{\Gamma(z,t)}\frac{\mu}{\sigma^2}\frac{K(z,t)}{D+K(z,t)}, \tag{32}$$

which represents the portfolio weight that would be chosen in the absence of presently-binding capital requirements, and

$$\frac{1}{\beta}\frac{K(z,t)-k\,VaR}{D+K(z,t)}, \tag{33}$$

the maximum feasible proportional allocation to risky assets under capital requirements. To simplify the interpretation of the optimal trading strategy, Figures 2 and 3 also plot the curves in equation (32) (lighter solid line) and (33) (dotted line).

Table 1 shows that for the set of parameters considered in Figures 2 and 3, the percentage VaR reported at the beginning of the first reporting period is 0.0348, which corresponds to a dollar VaR of 0.0698 (since $D + K^- = 2$). Therefore, the institution is to record an exception at the end of the first reporting period if capital drops below 1-0.0698=0.93002, or, equivalently, if total asset value drops below 1.93002. With this in mind, the figures display an intuitive pattern for the optimal portfolio allocation. For asset values sufficiently away from the point at which an additional exception is recorded, the allocation in the

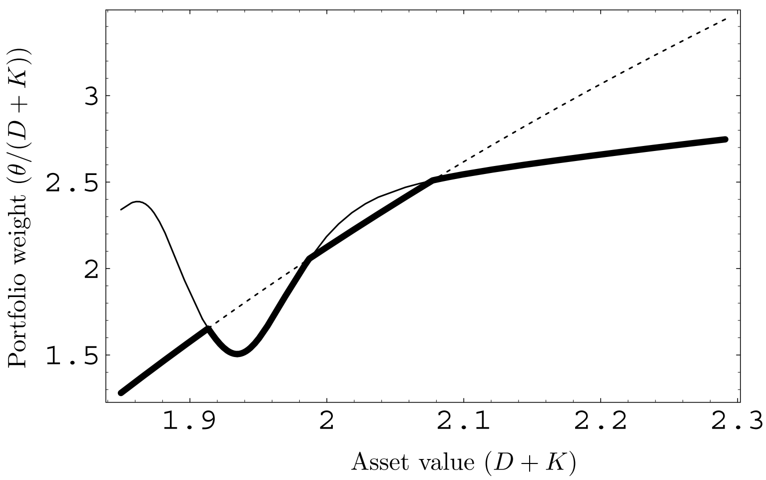

Figure 2: The graph plots the optimal portfolio allocation to risky assets at $t = .001$ (heavier solid line), the allocation that would be chosen in the absence of currently-binding capital requirements (lighter solid line) and the maximum feasible portfolio allocation (dotted line) as a function of asset value, for the case $\gamma = .25$, $\varepsilon = 0$, $\lambda = .01$, $\beta = .08$, $i = 0$, $k = 9.49$, $K^{-} = 1$ and $D = 1$.

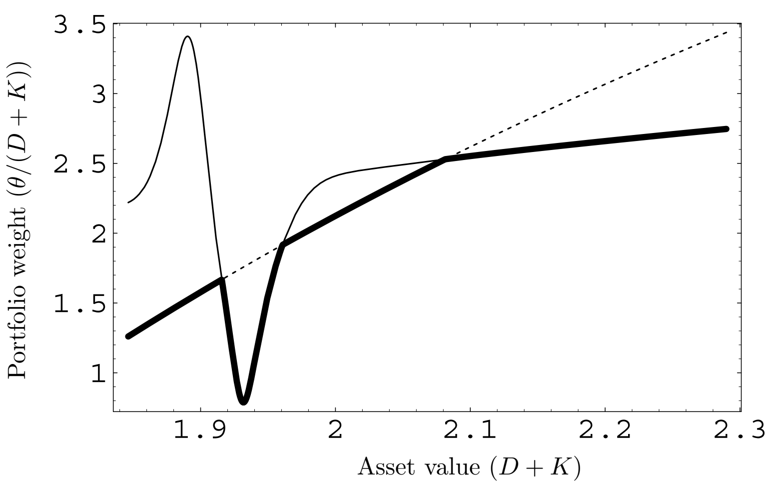

Figure 3: The graph plots the optimal portfolio allocation to risky assets at $t = .001$ (heavier solid line), the allocation that would be chosen in the absence of currently-binding capital requirements (lighter solid line) and the maximum feasible portfolio allocation (dotted line) as a function of asset value, for the case $\gamma = .25$, $\varepsilon = 0$, $\lambda = .01$, $\beta = .08$, $i = 0$, $k = 9.49$, $K^{-} = 1$ and $D = 1$.

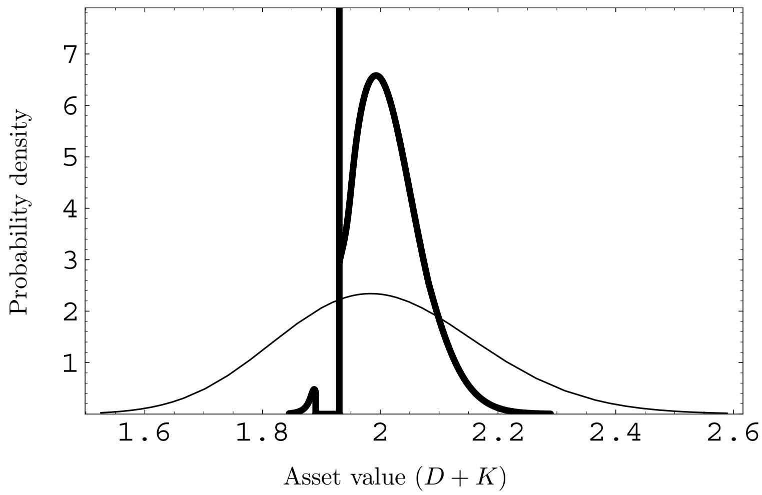

Figure 4: The graph shows the probability distribution of asset value at the end of the first reporting period ($t = .004$) in the presence of capital requirements (heavier solid line) and in the unconstrained model (lighter solid line), for the case $D = 1$, $K^- = 1$, $\beta = .08$, $\lambda = .01$, $\gamma = .25$, $\varepsilon = 0$, $i = 0$ and $k = 9.49$.

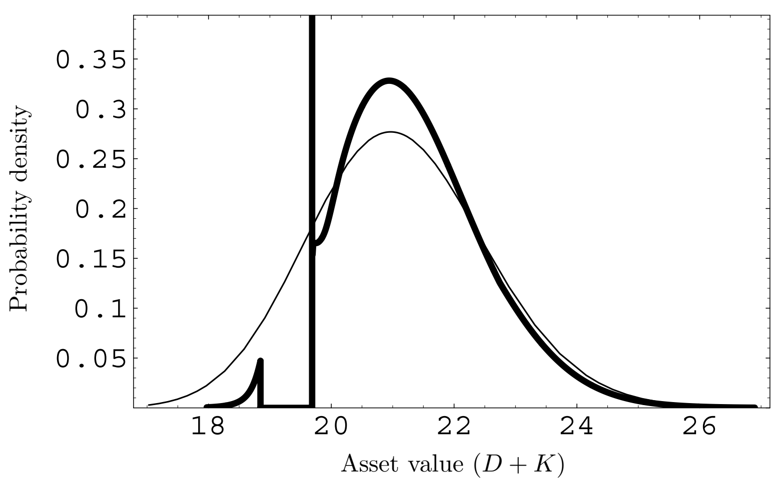

Figure 5: The graph shows the probability distribution of asset value at the end of the first reporting period ($t = .004$) in the presence of capital requirements (heavier solid line) and in the unconstrained model (lighter solid line), for the case $D = 1$, $K^- = 20$, $\beta = .08$, $\lambda = .01$, $\gamma = .25$, $\varepsilon = 0$, $i = 0$ and $k = 9.49$.

absence of binding capital requirements is close to $\frac{\mu}{\gamma\sigma^2} \times \frac{K}{D+K}$, (that is, between 2.226 and 2.756 for the range of asset values considered in the figures). However, this unconstrained allocation tends to increase significantly for asset values just below 1.93002 and to drop for asset values just above this level. This reflects the attempt on the part of the financial institution to avoid recording an exception at the end of the reporting period. In fact, at the end of the reporting period (that is, when $t = \tau^-$), the unconstrained allocation to stocks becomes discontinuous at this critical asset value, converging to 0 as asset value approaches this level from above and to $+\infty$ as asset value approaches this level from below. As shown in the figures, the institution optimally selects the reported VaR (and hence the location of the upper bound on the portfolio weight in equation (33)) so that no increase in risk-taking occurs for asset values around the critical level at which an exception is recorded. As a result, VaR-based capital requirements unambiguously decrease the financial institution risk-taking.

Changing the value of the current reserve multiple k or the number i of exceptions in the current backtesting period leaves the qualitative pattern in Figures 2 and 3 unchanged. An increase in k reduces the optimal portfolio allocation in the constrained model primarily by shifting down the upper bound in equation (33). An increase in the number i of violations in the current backtesting period primarily affects the extent of the drop in the unconstrained and constrained portfolio allocation for asset values around the critical level (since the cost of an additional exception depends on the corresponding increase $k(i+1)-k(i)$ in the reserve multiple over the next backtesting period). A change in the value of the pecuniary penalty coefficient λ has a similar effect, primarily affecting the cost of an additional exception and hence the extent of the drop in the portfolio allocation around the critical level.

Figure 4 shows the probability distribution of the asset value at the end of the reporting period ($t = \tau^-$), before the penalty for an exception is applied, for the same values of the parameters as in Figures 2 and 3. Figure 5 shows the probability distribution of the end-of-period asset values assuming that $K^- = e^3 \approx 20.09$ (which corresponds to an initial leverage ratio of about 0.05). For comparison purposes, Figures 4 and 5 also plot the probability density of end-of-period asset value in the unconstrained case discussed in Section 2 (lighter solid line).

The probability densities in the presence of capital requirements display a singularity at the critical asset value below which an exception is recorded. This follows from the behavior of the optimal portfolio allocation shown in Figures 2 and 3 and reflects the fact that in the presence of capital requirements the financial institution optimally chooses its investment strategy so that there is a positive probability that the end-of-period asset value will exactly equal the initial asset value minus the reported VaR. In addition, there is a zero probability of asset values just slightly below the critical level, which are clearly suboptimal. This follows from the fact that, as noted in Proposition 5, end-of-period capital equals $f(Z_{\nu^*}(\tau))$, where the function f (which is given in Appendix C) is discontinuous at the level of the state variable $Z_{\nu^*}(\tau)$ at which end-of-period capital is equal to the initial capital minus the reported VaR.

Parameter values: $\gamma = .25$, $\varepsilon = 0$, $\beta = .08$, $\lambda = .01$, $\tau = .004$, $m = 2$													
t	i	k	LR	v	Γ	$\iota^\top\bar{\pi}$	$\iota^\top\pi$	$v_{.90}$	$v_{.99}$	Γ^U	π^U	$v^U_{.90}$	$v^U_{.99}$
0	0	9.49	.05	.0675	.2838	4.092	3.898	.0675	.0675	.2357	4.927	.0853	.1488
0	0	9.49	.50	.0355	.2838	4.092	2.046	.0355	.0355	.0991	6.148	.1062	.1842
0	0	9.49	.95	.0034	.2838	4.092	0.194	.0034	.0034	.0086	6.698	.1155	.1994
0	0	12.65	.05	.0528	.3093	3.735	3.558	.0528	.0528	.2357	4.927	.0853	.1488
0	0	12.65	.50	.0277	.3093	3.735	1.867	.0277	.0277	.0991	6.148	.1062	.1842
0	0	12.65	.95	.0026	.3093	3.735	0.177	.0026	.0026	.0086	6.698	.1155	.1994
0	5	9.49	.05	.0681	.2895	4.017	3.826	.0681	.0681	.2357	4.927	.0853	.1488
0	5	9.49	.50	.0357	.2895	4.017	2.008	.0357	.0357	.0991	6.148	.1062	.1842
0	5	9.49	.95	.0034	.2895	4.017	0.191	.0034	.0034	.0086	6.698	.1155	.1994
0	5	12.65	.05	.0515	.2905	3.951	3.764	.0515	.0993	.2357	4.927	.0853	.1488
0	5	12.65	.50	.0270	.2905	3.951	1.976	.0270	.0520	.0991	6.148	.1062	.1842
0	5	12.65	.95	.0026	.2905	3.951	0.187	.0026	.0049	.0086	6.698	.1155	.1994
0	10	9.49	.05	.0667	.2770	4.191	3.992	.0667	.1018	.2357	4.927	.0853	.1488
0	10	9.49	.50	.0350	.2770	4.191	2.095	.0350	.0536	.0991	6.148	.1062	.1842
0	10	9.49	.95	.0033	.2770	4.191	0.199	.0033	.0051	.0086	6.698	.1155	.1994
0	10	12.65	.05	.0515	.2899	3.959	3.771	.0515	.0992	.2357	4.927	.0853	.1488
0	10	12.65	.50	.0270	.2899	3.959	1.979	.0270	.0527	.0991	6.148	.1062	.1842
0	10	12.65	.95	.0026	.2899	3.959	0.188	.0026	.0049	.0086	6.698	.1155	.1994
.996	0	9.49	.05	.0667	.2770	4.191	3.992	.0667	.1022	.2405	4.827	.0836	.1460
.996	0	9.49	.50	.0350	.2770	4.191	2.095	.0350	.0536	.0850	7.169	.1237	.2139
.996	0	9.49	.95	.0033	.2770	4.191	0.199	.0033	.0051	.0070	8.202	.1410	.2415
.996	0	12.65	.05	.0515	.2899	3.960	3.772	.0515	.0871	.2405	4.827	.0836	.1460
.996	0	12.65	.50	.0270	.2899	3.960	1.980	.0270	.0457	.0850	7.169	.1237	.2139
.996	0	12.65	.95	.0026	.2899	3.960	0.188	.0026	.0043	.0070	8.202	.1410	.2415
.996	5	9.49	.05	.0681	.2889	4.026	3.835	.0681	.0681	.2405	4.827	.0836	.1460
.996	5	9.49	.50	.0357	.2889	4.026	2.013	.0357	.0357	.0850	7.169	.1237	.2139
.996	5	9.49	.95	.0034	.2889	4.026	0.191	.0034	.0034	.0070	8.202	.1410	.2415
.996	5	12.65	.05	.0530	.3132	3.697	3.522	.0530	.0530	.2405	4.827	.0836	.1460
.996	5	12.65	.50	.0278	.3132	3.697	1.848	.0278	.0278	.0850	7.169	.1237	.2139
.996	5	12.65	.95	.0026	.3133	3.697	0.175	.0026	.0026	.0070	8.202	.1410	.2415
.996	10	9.49	.05	.0667	.2770	4.191	3.992	.0667	.1147	.2405	4.827	.0836	.1460
.996	10	9.49	.50	.0350	.2770	4.191	2.095	.0350	.0601	.0850	7.169	.1237	.2139
.996	10	9.49	.95	.0033	.2770	4.191	0.199	.0033	.0057	.0070	8.202	.1410	.2415
.996	10	12.65	.05	.0515	.2899	3.960	3.772	.0515	.0994	.2405	4.827	.0836	.1460
.996	10	12.65	.50	.0270	.2899	3.960	1.980	.0270	.0522	.0850	7.169	.1237	.2139
.996	10	12.65	.95	.0026	.2900	3.960	0.188	.0026	.0050	.0070	8.202	.1410	.2415

Table 5: The table shows the values of the reported normalized daily VaR (v), the RRA coefficient of the dual value function (Γ), the upper bound on the portfolio allocation to risky assets ($\iota^\top\bar{\pi}$), the portfolio allocation to risky assets ($\iota^\top\pi$), the true normalized 90% VaR ($v_{.90}$) and the true normalized 99% VaR ($v_{.99}$) in the presence of capital requirements, as well as the RRA coefficient (Γ^U), the total portfolio allocation to risky assets (π^U) and the true normalized 90% and 99% VaRs ($v^U_{.90}$ and $v^U_{.99}$) in the unconstrained case.

4.2 Two Risky Assets

To check how the results reported in the previous section are affected by the assumption of a single risky asset, we also compute the optimal reporting and investment strategies for the case of two risky assets. As mentioned at the beginning of this section, in order to allow a comparison with the case of a single risky asset, we choose the price coefficient μ and σ so that the risk premium and volatility of the mean-variance efficient portfolio of risky assets are the same as in previous examples. Finally, we assume that the two assets have the same risk weights, so that the risk weight for the market portfolio is also the same as in the previous examples.

Table 5 reports the results for the same set of parameters as in Table 1, but with $m = 2$. The ability to engage in asset substitution to limit the impact of capital requirements in the presence of multiple risky assets results in higher VaRs than in the case of a single risky asset. However, a comparison of the results in Table 5 with those in Table 1 shows that this effect is quite small. Moreover, all the qualitative features of the optimal reporting and trading strategy described in the previous subsection also apply in the presence of multiple risky assets. This is confirmed by Figure 6, which plots the total allocation to risky assets as a function of asset value at time $t = 0.003$ for the case $D = 1$, $K^- = 1$, $i = 0$ and $k = 9.49$. The total allocation to risky assets is very close to that displayed in Figure 3 for the case of a single risky asset.

Figure 7 shows the contribution of the two risky assets to the total portfolio allocation. As implied by Proposition 6, the optimal portfolio policy significantly diverges from two-fund separation. As the constraint increasingly binds (that is, as the total asset value decreases) the weight of the second asset (the one with the lower ratio of risk premium to risk coefficient) in the optimal risky asset portfolio is progressively reduced. If the total asset value at $t = 0.003$ is below 1.904, the second asset is entirely omitted from the optimal portfolio. However, in spite of this lack of portfolio separation, the total allocation to stocks $\pi_1 + \pi_2$ almost exactly matches that in Figure 2 for the case of a single risky asset.

5 Concluding Remarks

We study the dynamic investment and reporting problem of a financial institution subject to capital requirements based on self-reported VaR estimates, as in the Basel Committee's Internal Models Approach (IMA). We characterize the solution of this problem using martingale duality and parametric quadratic programming techniques. We find that capital requirements based on the IMA can be very effective in curbing portfolio risk and inducing truthful revelation of this risk. Even with constant price coefficients, optimal portfolios in the presence of capital requirements do not display two-fund separation: we show that as capital requirements become progressively more binding following losses, financial institutions find it optimal to rebalance their portfolios in favor of assets characterized by high expected returns (high systematic risks) relative to the regulatory risk weights. However, optimal portfolios satisfy a local three-fund separation property, the three funds being the riskless asset, the mean-variance efficient portfolio of risky assets and a risk-weight-constrained minimum-variance portfolio of risky assets. For no choice of the parameters we find IMA-based capital requirements leading to an increased probability of extreme losses.

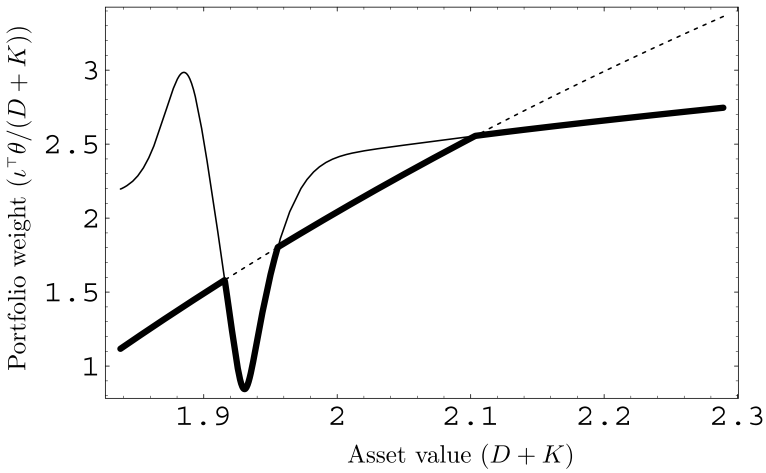

Figure 6: The graph plots the optimal portfolio allocation to risky assets at $t = .003$ (heavier solid line), the allocation that would be chosen in the absence of currently-binding capital requirements (lighter solid line) and the maximum feasible portfolio allocation (dotted line) as a function of asset value, for the case $\gamma = .25$, $\varepsilon = 0$, $\lambda = .01$, $\beta = .08$, $i = 0$, $k = 9.49$, $K^- = 1$ and $D = 1$.

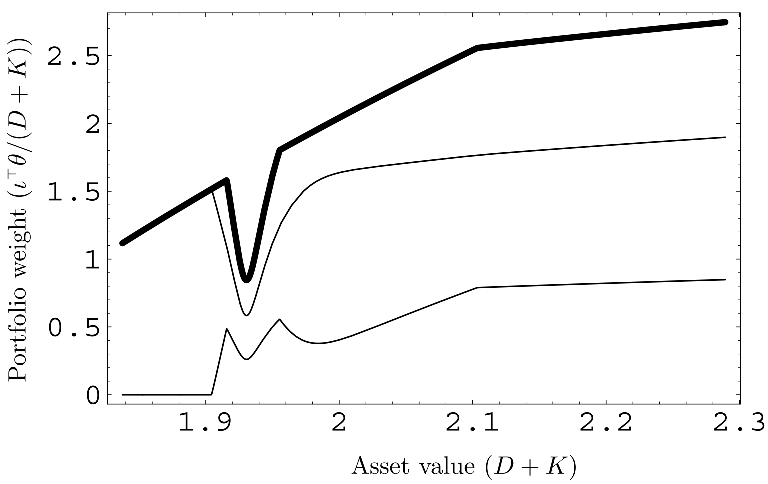

Figure 7: The graph plots the optimal portfolio allocation to risky assets at $t = .003$ (heavier solid line) and to each of the two risky assets (lighter solid lines) as a function of asset value, for the case $\gamma = .25$, $\varepsilon = 0$, $\lambda = .01$, $\beta = .08$, $i = 0$, $k = 9.49$, $K^- = 1$ and $D = 1$.

Appendix A:
The Pre-Commitment Approach

While the introduction of the internal models approach responded to the industry's request that financial institutions be allowed to use in-house models for measuring market risk, its implementation has been criticized on at least four counts. First, the choice of a multiplier is arbitrary and has been questioned as being too high and thus representing a disincentive for financial institutions to accurately measure and report the risk of their trading portfolios (Elderfield (1995)). Second, the "square root of ten" rule generally used to convert the daily VaR measures used for benchmarking into the two-week measures used to determine capital requirements is imprecise if the return distribution is not normal (Danielsson, Hartmann and de Vries (1998)). Third, backtesting is likely to have limited power in detecting inaccurate risk estimates (Kupiec (1995)). Fourth, the approach relies on a "building-block" procedure, in which capital requirements to cover general market risk are estimated separately from capital requirements to cover credit (or idiosyncratic) risk (Hendricks and Hirtle (1997)).

These criticisms have induced the U.S. Federal Reserve Bank to evaluate an alternative approach to the determination of capital reserves to cover market risk. This approach, known as the "Pre-Commitment Approach" (PCA), requires financial institutions to report at the beginning of every quarter their estimated maximum trading loss over the quarter (that is, a 100% quarterly VaR). The minimum capital requirement over the quarter equals this reported maximum loss. The observation of an exception to the reported maximum loss at the end of any quarter would result in pecuniary sanctions or other corrective actions. While the pre-commitment approach to capital regulation has not been fully implemented, a pilot run involving ten major international banks took place in 1996-1997.[38]

The resulting optimal reporting and investment problem is a special case of the problem in (14) with $k(i) = 1$ for all i and $\beta = 0$. Thus, the solution of this problem can be obtained as described in Section 3. The next proposition provides an explicit solution to this problem when $\varepsilon = 0$ and $r = 0$.[39,40] We start with a preliminary remark.

Remark 4 *If $u(0) = -\infty$, default is never optimal and hence capital at the end of any reporting period must be larger than $\frac{\lambda(K^- - VaR)}{1+\lambda}$ (see Remark 2). If $r = 0$, the same must be true for capital at any time during the reporting period. Since with $\beta = 0$ and $k(i) = k$ for all i reporting a VaR smaller than the minimum possible level of capital in the reporting period divided by k is never optimal, we must have $k\,VaR \geq \frac{\lambda(K^- - VaR)}{1+\lambda}$, or*

$$VaR \geq \frac{\lambda K^-}{\lambda + (1+\lambda)k}$$

in this case.

[38] See Considine (1998) for details.

[39] More generally, the results in Proposition 7 would be applicable even with a non-zero interest rate if the financial institution were required to maintain its capital above a multiple of the *discounted value* of the reported maximum loss, that is, if the constraint $K_t \geq k\,VaR$ for $t \in [(h-1)\tau, h\tau)$ in (14) were replaced by the constraint $K_t \geq k\,VaRe^{-r(h\tau - t)}$.

[40] When $\beta = 0$ and $r > 0$, the investment problem in (14) becomes a singular control problem whose dual value function can be characterized using the approach of He and Pagès (1993). We do not pursue this characterization since the difference from the explicit solution for the case $r = 0$ is likely to be minor.

Proposition 7 *Suppose $\varepsilon = 0$, $r = 0$, $\beta = 0$ and $k(i) = k$ for all i. Then for*

$$\frac{\lambda K^-}{\lambda + (1+\lambda)k} 1_{\{\gamma > 1\}} \leq VaR \leq \frac{K^-}{1+k},$$

the value function $V^P(K, K^-, VaR, t)$ for the problem in (14) satisfies

$$V^P(K, K^-, VaR, t) = \min_{\psi \geq 0} [\tilde{V}^P(\psi, K^-, VaR, t) + \psi K],$$

where for $t \in [(h-1)\tau, h\tau)$

$$\begin{aligned}
&\tilde{V}^P(z, K^-, VaR, t) \\
= \quad & -\frac{\alpha_h}{b}\left(\frac{z}{\alpha_h}\right)^b e^{-b\frac{|\kappa|^2}{2\gamma}(h\tau - t)} N\left(\frac{\log\left(\frac{\alpha_h z_1}{z}\right) + \frac{2-\gamma}{2\gamma}|\kappa|^2(h\tau - t)}{|\kappa|\sqrt{h\tau - t}}\right) \\
& + \alpha_h u(K^- - VaR) \\
& \quad \times \left(N\left(\frac{\log\left(\frac{\alpha_h z_2}{z}\right) + \frac{1}{2}|\kappa|^2(h\tau - t)}{|\kappa|\sqrt{h\tau - t}}\right) - N\left(\frac{\log\left(\frac{\alpha_h z_1}{z}\right) + \frac{1}{2}|\kappa|^2(h\tau - t)}{|\kappa|\sqrt{h\tau - t}}\right)\right) \\
& - (K^- - VaR)z \\
& \quad \times \left(N\left(\frac{\log\left(\frac{\alpha_h z_2}{z}\right) - \frac{1}{2}|\kappa|^2(h\tau - t)}{|\kappa|\sqrt{h\tau - t}}\right) - N\left(\frac{\log\left(\frac{\alpha_h z_1}{z}\right) - \frac{1}{2}|\kappa|^2(h\tau - t)}{|\kappa|\sqrt{h\tau - t}}\right)\right) \\
& - \frac{\alpha_h}{b}\left(\frac{z}{\alpha_h(1+\lambda)}\right)^b e^{-b\frac{|\kappa|^2}{2\gamma}(h\tau - t)} \\
& \quad \times \left(N\left(\frac{\log\left(\frac{\alpha_h z_3}{z}\right) + \frac{2-\gamma}{2\gamma}|\kappa|^2(h\tau - t)}{|\kappa|\sqrt{h\tau - t}}\right) - N\left(\frac{\log\left(\frac{\alpha_h z_2}{z}\right) + \frac{2-\gamma}{2\gamma}|\kappa|^2(h\tau - t)}{|\kappa|\sqrt{h\tau - t}}\right)\right) \\
& - \frac{\lambda(K^- - VaR)}{1+\lambda} z \\
& \quad \times \left(N\left(\frac{\log\left(\frac{\alpha_h z_3}{z}\right) - \frac{1}{2}|\kappa|^2(h\tau - t)}{|\kappa|\sqrt{h\tau - t}}\right) - N\left(\frac{\log\left(\frac{\alpha_h z_2}{z}\right) - \frac{1}{2}|\kappa|^2(h\tau - t)}{|\kappa|\sqrt{h\tau - t}}\right)\right) \\
& + \alpha_h u((\lambda(1+k) + k)\,VaR - \lambda K^-) N\left(-\frac{\log\left(\frac{\alpha_h z_3}{z}\right) + \frac{1}{2}|\kappa|^2(h\tau - t)}{|\kappa|\sqrt{h\tau - t}}\right) \\
& - k\,VaR\,z\,N\left(-\frac{\log\left(\frac{\alpha_h z_3}{z}\right) - \frac{1}{2}|\kappa|^2(h\tau - t)}{|\kappa|\sqrt{h\tau - t}}\right),
\end{aligned}$$

$$z_1 = (K^- - VaR)^{-\gamma},$$

$$z_2 = \min((1+\lambda)z_1, z_3),$$

$$z_3 = \begin{cases} \frac{u(K^- - VaR)}{K^- - (1+k)\,VaR} & \text{if } 0 \leq VaR \leq \frac{(\lambda - (1+\lambda)\gamma)^+ K^-}{(\lambda-(1+\lambda)\gamma)^+(1+k)+k}, \\ (1+\lambda)\left(\frac{1-\gamma}{\gamma}(\lambda K^- - (\lambda + (1+\lambda)k)\,VaR)\right)^{-\gamma} & \text{if } \frac{(\lambda-(1+\lambda)\gamma)^+ K^-}{(\lambda-(1+\lambda)\gamma)^+(1+k)+k} < VaR \leq \frac{\lambda K^-}{\lambda(1+k)+k}, \\ (1+\lambda)\left((\lambda + (1+\lambda)k)\,VaR - \lambda K^-\right)^{-\gamma} & \text{if } \frac{\lambda K^-}{\lambda(1+k)+k} < VaR \leq \frac{K^-}{1+k}, \end{cases}$$

Parameter values: $\varepsilon = 0$, $\beta = 0$, $k = 1$, $\tau = .25$, $t = 0$													
γ	λ	LR	v	Γ	π	$v_{.90}$	$v_{.99}$	$v_{1.0}$	Γ^U	π^U	$v^U_{.90}$	$v^U_{.99}$	$v^U_{1.0}$
.25	.005	.05	.3104	.2551	4.551	.4992	.6422	.6422	.2357	4.927	.5404	.7508	1.000
.25	.005	.50	.1629	.2551	2.389	.2620	.3371	.3371	.0991	6.148	.6616	.8626	1.000
.25	.005	.95	.0155	.2551	0.227	.0249	.0320	.0320	.0086	6.698	.7051	.7889	1.000
.25	.010	.05	.3536	.2591	4.482	.4914	.5990	.5990	.2357	4.927	.5404	.7508	1.000
.25	.010	.50	.1856	.2591	2.353	.2579	.3144	.3144	.0991	6.148	.6616	.8626	1.000
.25	.010	.95	.0176	.2591	0.223	.0245	.0298	.0298	.0086	6.698	.7051	.8914	1.000
.25	.050	.05	.4763	.2719	4.271	.4763	.4763	.4763	.2357	4.927	.5404	.7508	1.000
.25	.050	.50	.2500	.2719	2.242	.2500	.2500	.2500	.0991	6.148	.6616	.8626	1.000
.25	.050	.95	.0237	.2719	0.213	.0237	.0237	.0237	.0086	6.698	.7051	.8914	1.000
.50	.005	.05	.4317	.5009	2.318	.2772	.4371	.5208	.5000	2.323	.2771	.4422	1.000
.50	.005	.50	.2266	.5009	1.217	.1455	.2294	.2734	.2078	2.933	.3624	.6022	1.000
.50	.005	.95	.0215	.5009	0.115	.0138	.0169	.0259	.0149	3.892	.4733	.7232	1.000
.50	.010	.05	.4480	.5013	2.316	.2772	.4422	.5046	.5000	2.323	.2771	.4422	1.000
.50	.010	.50	.2351	.5013	1.216	.1455	.2321	.2649	.2078	2.933	.3624	.6022	1.000
.50	.010	.95	.0223	.5013	0.115	.0138	.0220	.0251	.0149	3.892	.4733	.7232	1.000
.50	.050	.05	.4763	.5018	2.314	.2773	.4423	.4763	.5000	2.323	.2771	.4422	1.000
.50	.050	.50	.2500	.5018	1.215	.1455	.2321	.2500	.2078	2.933	.3624	.6022	1.000
.50	.050	.95	.0237	.5018	0.115	.0138	.0220	.0237	.0149	3.892	.4733	.7232	1.000

Table 6: The table shows the values of the reported normalized daily VaR (v), the RRA coefficient of the dual value function (Γ), the upper bound on the portfolio weight ($\bar{\pi}$), the portfolio weight (π), the true normalized 90% VaR ($v_{.90}$), the true normalized 99% VaR ($v_{.99}$) and the true 100% VaR ($v_{1.0}$) in the presence of capital requirements determined according to the PCA. For comparison purposes, the table also reports the RRA coefficient (Γ^U), the portfolio weight (π^U) and the true normalized 90%, 99% and 100% VaRs ($v^U_{.90}$, $v^U_{.99}$ and $v^U_{1.0}$) in the unconstrained case.

$$\alpha_h = \left((1-\gamma) \max_{v \geq 0} \min_{\psi \geq 0} \left[\tilde{V}^P(\psi, 1, v, T-\tau) + \psi \right] \right)^{nT-h} \tag{34}$$

and N is the standard normal distribution function

Table 6 illustrates the impact of capital requirements determined on the basis of the pre-commitment approach. For the same set of parameters considered in Section 4 (but with $\tau = 1/4$ to correspond to quarterly reporting), the table shows the reported VaR, as well as the value function CRRA coefficient and the true 90%, 99% and 100% VaRs in both the constrained and the unconstrained model. Since time effects are negligible in the pre-commitment approach, only the values at $t = 0$ are reported. We find that capital requirements determined according to the pre-commitment approach are also effective in curbing excessive risk taking. Since over-reporting a 100% VaR is never optimal (because truthful reporting is already sufficient to avoid all penalties with probability one), reported VaRs are generally below true 100% VaRs. However, as the penalty coefficient λ increases, reported VaRs converge quickly to truthful reporting. For the parameters in Table 6 a penalty of 5% is sufficient to achieve this. Since the maximum possible period loss under PCA capital requirements equals $K^{-} - VaR$, the reported VaR in case of truthful reporting must be such that $VaR = K^{-} - VaR$, or $v = (1 - \mathrm{LR})/2$. This corresponds to the values of .4763, .2500 and .0237 for the three leverage ratios in Table 6. Clearly, further increases in the penalty coefficient λ beyond 5% would have no additional impact on reported VaRs or on optimal trading strategies (since pecuniary penalties are already entirely avoided).

A comparison of Table 6 with Tables 1–4 shows that risk-taking tends to be higher under capital requirements determined according to the PCA than under capital requirements determined according to the IMA. This is consistent with the view expressed by the institutions participating in the 1996-97 pilot run of the pre-commitment approach that "pre-committed capital amounts were significantly less than market risk capital requirements estimated to apply under [the 1996 Amendment]" and that "the '3X' multiplier, as well as the specific risk component [in the internal models approach] lead to excessive regulatory capital requirements for their trading positions" (Considine (1998), p. 134).

Appendix B:
Proofs

Proof of Lemma 1: Letting

$$g(z) = -\frac{z^b}{b} + (\varepsilon - D)z - \frac{\varepsilon^{1-\gamma}}{1-\gamma}, \tag{35}$$

it is easily verified that under the stated assumptions

$$\lim_{z\to 0} g(z) > 0$$

and

$$g(\varepsilon^{-\gamma}) = -D\varepsilon^{-\gamma} \leq 0.$$

In addition,

$$g'(z) = -z^{-\frac{1}{\gamma}} + (\varepsilon - D) < 0 \quad \text{for } z < \varepsilon^{-\gamma}.$$

The existence of a unique $z_U \in (0, \varepsilon^{-\gamma}]$ solving equation (9) then immediately follows from the continuity and monotonicity of g. □

Proof of Lemma 2: Suppose first that $u(0) > -\infty$. Let

$$K_1^* = \arg\max_{K\geq 0}\,[u(K) - zK] = \arg\max_{K\geq 0}\left[\frac{(K+\varepsilon)^{1-\gamma}}{1-\gamma} - zK\right] = \left(z^{-\frac{1}{\gamma}} - \varepsilon\right)^+$$

and

$$K_2^* = \arg\max_{-D\leq K\leq 0}\,[u(K) - zK] = \arg\max_{-D\leq K\leq 0}\left[\frac{\varepsilon^{1-\gamma}}{1-\gamma} - zK\right] = -D,$$

so that the solution K^* of the maximization problem in (8) equals either K_1^* or K_2^*.

If $z \leq z_U$, then

$$K_1^* = z^{-\frac{1}{\gamma}} - \varepsilon \tag{36}$$

and

$$u(K_1^*) - zK_1^* = -\frac{z^b}{b} + \varepsilon z \geq \frac{\varepsilon^{1-\gamma}}{1-\gamma} + Dz = u(K_2^*) - zK_2^*, \tag{37}$$

where the equality in (36) follows from the fact that $z \leq z_U \leq \varepsilon^{-\gamma}$, while the inequality in (37) follows from the fact that, letting g be the function in (35), we have $g(z) > 0$ (since $z < z_U$, $g(z_U) = 0$ and g is monotonically decreasing for $z < \varepsilon^{1-\gamma}$). Hence, $K^* = K_1^*$.

On the other hand, if $z_U < z < \varepsilon^{-\gamma}$, then the argument in the previous paragraph implies that (36) still holds, but the inequality in (37) is reversed. Hence, $K^* = K_2^*$.

Finally, if $z > \varepsilon^{-\gamma}$, then $K_1^* = 0$ and hence

$$u(K_1^*) - zK_1^* = \frac{\varepsilon^{1-\gamma}}{1-\gamma} \leq \frac{\varepsilon^{1-\gamma}}{1-\gamma} + Dz = u(K_2^*) - zK_2^*.$$

Therefore, $K^* = K_2^*$.

Next, suppose that $u(0) = -\infty$. Then $u(K) = -\infty$ for all $K \leq 0$ and hence the solution of (8) must be given by $K^* = K_1^*$ for all $z > 0$. Moreover, K_1^* is strictly positive in this case.

The properties of $f^U(z)$ and $\tilde{v}^U(z)$ can be directly verified from their definitions. $\square$

Proof of Proposition 1: The expression for $\tilde{V}^U$ follows from direct integration, using the fact that

$$\tilde{V}^U(z,t) = \mathrm{E}_t\left[\tilde{v}^U\left(ze^{-\left(r+\frac{|\kappa|^2}{2}\right)(T-t)-\kappa^\top(w(T)-w(t))}\right)\right], \tag{38}$$

where $\tilde{v}^U$ is the function in (11). The fact that $\tilde{V}$ is strictly convex for $t < T$ can be verified by differentiating the expression in (12) or by noticing that (38) implies

$$\begin{aligned}\tilde{V}_z^U(z,t) &= \mathrm{E}_t\left[e^{-\left(r+\frac{|\kappa|^2}{2}\right)(T-t)-\kappa^\top(w(T)-w(t))}\tilde{v}_z^U\left(ze^{-\left(r+\frac{|\kappa|^2}{2}\right)(T-t)-\kappa^\top(w(T)-w(t))}\right)\right] \\ &= -\mathrm{E}_t\left[e^{-\left(r+\frac{|\kappa|^2}{2}\right)(T-t)-\kappa^\top(w(T)-w(t))}f^U\left(ze^{-\left(r+\frac{|\kappa|^2}{2}\right)(T-t)-\kappa^\top(w(T)-w(t))}\right)\right]\end{aligned} \tag{39}$$

(where $\tilde{v}_z^U$ denotes the right-derivative of $\tilde{v}^U$) and that f^U is decreasing for $z > 0$ and strictly decreasing for $z < z_U$ (see Lemma 2). $\square$

Proof of Proposition 2: The equality $K^U(t) = -\tilde{V}_z^U(Z(t),t)$ follows from the fact that

$$K^U(t) = \mathrm{E}_t\left[\frac{Z(T)}{Z(t)}f^U(Z(T))\right]$$

and (39). The expression for θ can then be verified by applying Itô's lemma to $-\tilde{V}_z^U(Z(t),t)$. $\square$

Proof of Corollary 1: The results follow immediately from Proposition 2 and the expression for the dual value function in Proposition 1. $\square$

Proof of Proposition 3: Letting $\alpha = K - \theta^\top \bar{1}$, the constraint in (14) is equivalent to $(\alpha_t, \theta_t) \in A$, where

$$A = \{(\alpha,\theta) \in \mathrm{I\!R} \times \mathrm{I\!R}^m : \alpha \geq k\,VaR - (\bar{1}-\beta)^\top\theta^+ + (\bar{1}+\beta)^\top\theta^-\}.$$

For $\nu = (\nu_0, \nu_-) \in \mathrm{I\!R} \times \mathrm{I\!R}^m$, let $\delta_A(\nu) = \sup_{(\alpha,\theta)\in A}[-(\alpha\nu_0 + \theta\nu_-)]$ denote the support function of $-A$ and let $\tilde{A} = \{\nu \in \mathrm{I\!R}^{m+1} : \delta_A(\nu) < +\infty\}$ denote its dual cone. It is easily verified that

$$\tilde{A} = \{(\nu_0,\nu_-) \in \mathrm{I\!R} \times \mathrm{I\!R}^m : \nu_0 \geq 0,\ \nu_0(\bar{1}-\beta) \leq \nu_- \leq \nu_0(\bar{1}+\beta)\},$$

and

$$\delta_A(\nu) = -k\,VaR\;\nu_0 \quad \text{for } \nu \in \tilde{A}.$$

The claim then follows from Proposition 1 in Cuoco (1997).[41] □

Proof of Proposition 4: The monotonicity and convexity of V follow from the fact that (21) implies

$$\begin{aligned}\tilde{V}_z(z,t) \;=\; \mathrm{E}_t\Bigg[& e^{-\int_t^{h\tau}\left(r+\nu^*(u)+\frac{1}{2}\kappa_{\nu^*}(u)^2\right)du-\int_t^{h\tau}\kappa_{\nu^*}(u)\,dw(u)} \\ & \times\tilde{v}_z\left(ze^{-\int_t^{h\tau}\left(r+\nu^*(u)+\frac{1}{2}\kappa_{\nu^*}(u)^2\right)du-\int_t^{h\tau}\kappa_{\nu^*}(u)\,dw(u)}\right) \\ & -k\,VaR\int_t^{h\tau}e^{-\int_t^{u}\left(r+\nu^*(s)+\frac{1}{2}\kappa_{\nu^*}(s)^2\right)ds-\int_t^{u}\kappa_{\nu^*}(s)\,dw(s)}\nu^*(u)\,du\Bigg]\end{aligned} \tag{40}$$

and the fact that $\tilde{v}$ is strictly decreasing and convex in z (see Appendix C).

The PDE (22) is the HJB equation associated with the minimization problem in (21). The bounds on ν^* follow from the fact that, letting $\zeta = \nu_- - \nu_0\bar{1}$, we can write the minimization problem in (22) as

$$\min_{\substack{\nu_0 \geq 0 \\ -\nu_0\beta \leq \zeta \leq \nu_0\beta}} \left[\frac{1}{2}|\sigma^{-1}(\mu+\zeta)|^2 - \frac{\tilde{V}_z + k\,VaR}{z\tilde{V}_{zz}}\nu_0\right]. \tag{41}$$

The unconstrained solution to (41) is given by $\zeta^* = -\mu$, which is feasible in (41) as long as $\nu_0 \geq M = \max\{|\iota_i^\top\mu|/\iota_i^\top\beta : i = 1,\ldots,m\}$. Since $K = -\tilde{V}_z$, the constraint in (14) implies

$$\tilde{V}_z + k\,VaR \leq -\beta^\top(\theta^+ + \theta^-) \leq 0.$$

Hence, the term multiplying ν_0 in (41) is nonnegative. Thus, taking $\nu_0 > M$ in (41) increases the second term while leaving the first unchanged. Hence, $\nu_0^* \leq M$. The bounds on ν_-^* then follow immediately from the definition of $\tilde{A}$. □

Proof of Proposition 5: The proof of Theorem 1 in Cuoco (1997) shows that

$$K(t) = \mathrm{E}\left[-\frac{Z_{\nu^*}(h\tau)}{Z_{\nu^*}(t)}\tilde{v}_z\left(Z_{\nu^*}(h\tau)\right) + k\,VaR\int_t^{h\tau}\frac{Z_{\nu^*}(u)}{Z_{\nu^*}(t)}\nu^*(u)\,du \;\Big|\; \mathcal{F}_t\right].$$

Equation (24) then follows from (40), while equation (23) follows from an application of Itô's lemma to (24). □

Proof of Proposition 6: Letting $\zeta = \nu_- - \nu_0\bar{1}$, we can write the minimization problem in (22) as

$$\min_{\substack{\nu_0 \geq 0 \\ -\nu_0\beta \leq \zeta \leq \nu_0\beta}} \left[\frac{1}{2}|\sigma^{-1}(\mu+\zeta)|^2 + h\nu_0\right], \tag{42}$$

[41] It can be easily verified that the proof of Proposition 1 in Cuoco (1997) remains valid for the problem in (14) in spite of the fact that $v(K, K^-, VaR, i, k, h\tau)$ has a discontinuity at $K = K^- - VaR$.

where

$$h = -\frac{\tilde{V}_z + k\,VaR}{z^2 \tilde{V}_{zz}}.$$

Clearly, the solution (ν_0^*, ζ^*) to (42) is going to depend on the value of h. We will verify below that ν_0^* is a monotonically decreasing function of h. Since the constraints on ζ become less binding as ν_0^* increases, a constraint that does not bind for a given value of h will never bind for lower values of h. Clearly, when h is very large (ν_0^* is close to 0) exactly one constraint will bind for each component of ζ and the above argument implies that the other constraint will never bind. Let H be the $m \times m$ diagonal matrix whose i-th diagonal element is equal to $+1$ (respectively, -1) if the lower bound (respectively, the upper bound) binds on the i-th component of ζ for h sufficiently large. We can then rewrite the constraint in (42) as

$$H\zeta + \nu_0 \beta \geq 0.$$

The Lagrangian for this problem is

$$L = \frac{1}{2}(\mu + \zeta)^\top (\sigma\sigma^\top)^{-1}(\mu + \zeta) + h\nu_0 - \lambda^\top (H\zeta + \nu_0\beta),$$

where λ is a vector of Lagrangian multipliers.

Now suppose that for a given value of h only the first $i \leq m$ constraints on ζ bind. Then the last $m - i$ components of λ equal 0 (that is, $\lambda = I_i^\top I_i \lambda$) and the first-order order conditions for a minimum are

$$(\sigma\sigma^\top)^{-1}(\mu + \zeta^*) - H\lambda = 0, \tag{43}$$

$$I_i(H\zeta^* + \nu_0^*\beta) = 0, \tag{44}$$

$$h - \lambda^\top \beta \geq 0, \tag{45}$$

$$(h - \lambda^\top \beta)\nu_0^* = 0 \tag{46}$$

From equation (43)

$$\zeta^* = -\mu + \sigma\sigma^\top H\lambda \tag{47}$$

and hence, from equation (44) and the fact that $\lambda = I_i^\top I_i \lambda$

$$\begin{aligned}
\lambda &= I_i^\top (I_i H \sigma\sigma^\top H I_i^\top)^{-1} I_i (H\mu - \nu_0^*\beta) \\
&= I_i^\top I_i H H I_i^\top (I_i H I_i^\top I_i \sigma\sigma^\top I_i^\top I_i H I_i^\top)^{-1} I_i H H (H\mu - \nu_0^*\beta) \\
&= I_i^\top I_i H I_i^\top I_i H I_i^\top (I_i H I_i^\top)^{-1} (I_i \sigma\sigma^\top I_i^\top)^{-1} (I_i H I_i^\top)^{-1} I_i H I_i^\top I_i H (H\mu - \nu_0^*\beta) \\
&= H I_i^\top (I_i \sigma\sigma^\top I_i^\top)^{-1} I_i (\mu - \nu_0^* H\beta).
\end{aligned} \tag{48}$$

Substituting the above expression for λ in (47) gives

$$\zeta^* = -\mu + \sigma\sigma^\top I_i^\top (I_i \sigma\sigma^\top I_i^\top)^{-1} I_i (\mu - \nu_0^* H\beta), \tag{49}$$

while substituting in (45) and using (46) gives

$$\nu_0^* = \left(\frac{\beta^\top H I_i^\top (I_i \sigma\sigma^\top I_i^\top)^{-1} I_i \mu - h}{\beta^\top H I_i^\top (I_i \sigma\sigma^\top I_i^\top)^{-1} I_i H\beta} \right)^+. \tag{50}$$

As already noted, it is clear from (42) that there must exist some constant h_m such that ν_0^* is close to zero when $h > h_m$ and hence the constraint on each component of ζ binds (that, is all the components of λ are strictly positive). The above analysis (with $i = m$) then implies

$$\lambda = H(\sigma\sigma^\top)^{-1}(\mu - \nu_0^* H\beta),$$
$$\zeta^* = -\nu_0^* H\beta$$
$$\nu_0^* = \left(\frac{\beta^\top H(\sigma\sigma^\top)^{-1}\mu - h}{\beta^\top H(\sigma\sigma^\top)^{-1}H\beta}\right)^+$$

for $h > h_m$. Let

$$h_{m+1} = \beta^\top H(\sigma\sigma^\top)^{-1}\mu > h_m.$$

For $h \geq h_{m+1}$, $\nu_0^* = 0$, $\zeta^* = 0$ and $\lambda = H(\sigma\sigma^\top)^{-1}\mu$. For all the components of λ to be strictly positive, we must have

$$H = \operatorname{diag}\left(\operatorname{sign}\left((\sigma\sigma^\top)^{-1}\mu\right)\right).$$

On the other hand, for $h_m < h < h_{m+1}$ the strict positivity of λ amounts to

$$\nu_0^* < \min\left\{\eta_{m,j} : \eta_{m,j} > 0, j = 1, 2, \ldots, m\right\}, \tag{51}$$

where

$$\eta_{m,j} = \frac{\iota_j^\top H(\sigma\sigma^\top)^{-1}\mu}{\iota_j^\top H(\sigma\sigma^\top)^{-1}H\beta}.$$

The fact that $H(\sigma\sigma^\top)^{-1}H$ is positive definite and $\beta \in \mathbb{R}^m_{++}$ implies that there must be at least one $\eta_{m,j} > 0$, so that the minimum in (51) is well defined. By resorting the assets if needed, we can assume without loss of generality that the minimum is attained by $\eta_{m,m}$. The condition on ν_0^* is then equivalent to

$$h > \beta^\top H(\sigma\sigma^\top)^{-1}(\mu - \eta_{m,m}H\beta).$$

Thus,

$$h_m = \beta^\top H(\sigma\sigma^\top)^{-1}(\mu - \eta_{m,m}H\beta).$$

When $h \leq h_m$, the constraint on the last component of ζ no longer binds and there must exist some constant $h_{m-1} \leq h_m$ such that for $h_{m-1} < h \leq h_m$ the constraint on each of the first $m-1$ components of ζ binds. Since ν_0^* must be continuous at h_m (see, e.g., Theorem 1 in Tøndel, Johansen and Bemporad (2003)) and hence strictly positive, for these values of h we must have (from equations (48)–(50) with $i = m-1$)

$$\nu_0^* = \frac{\beta^\top H I_{m-1}^\top (I_{m-1}\sigma\sigma^\top I_{m-1}^\top)^{-1} I_{m-1}\mu - h}{\beta^\top H I_{m-1}^\top (I_{m-1}\sigma\sigma^\top I_{m-1}^\top)^{-1} I_{m-1} H\beta},$$
$$\zeta^* = -\mu + \sigma\sigma^\top I_{m-1}^\top (I_{m-1}\sigma\sigma^\top I_{m-1}^\top)^{-1} I_{m-1}(\mu - \nu_0^* H\beta),$$
$$\lambda = H I_{m-1}^\top (I_{m-1}\sigma\sigma^\top I_{m-1}^\top)^{-1} I_{m-1}(\mu - \nu_0^* H\beta).$$

The strict positivity of the first $m-1$ components of λ amounts to

$$\nu_0^* < \min\Big\{\eta_{m-1,j} : \eta_{m-1,j} > 0, j = 1, 2, \ldots, m-1\Big\}, \tag{52}$$

where

$$\eta_{m-1,j} = \frac{\iota_j^\top H I_{m-1}^\top (I_{m-1}\sigma\sigma^\top I_{m-1}^\top)^{-1} I_{m-1}\mu}{\iota_j^\top H I_{m-1}^\top (I_{m-1}\sigma\sigma^\top I_{m-1}^\top)^{-1} I_{m-1} H\beta}.$$

The fact that $I_{m-1}\lambda$ is continuous (see, e.g., Theorem 1 in Tøndel *et alii* (2003)) and hence strictly positive at h_m, so that

$$I_{m-1} H I_{m-1}^\top (I_{m-1}\sigma\sigma^\top I_{m-1}^\top)^{-1} I_{m-1}\mu > \eta_{m,m} I_{m-1} H I_{m-1}^\top (I_{m-1}\sigma\sigma^\top I_{m-1}^\top)^{-1} I_{m-1} H\beta,$$

$H I_{m-1}^\top (I_{m-1}\sigma\sigma^\top I_{m-1}^\top)^{-1} I_{m-1} H$ is positive definite and $\beta \in \mathbb{R}^m_{++}$ implies that there must be at least one $\eta_{m-1,j} > 0$. Without loss of generality, we can assume that the minimum in (52) is attained by $\eta_{m-1,m-1}$. The condition on ν_0^* is then equivalent to

$$h > \beta^\top H I_{m-1}^\top (I_{m-1}\sigma\sigma^\top I_{m-1}^\top)^{-1} I_{m-1}(\mu - \eta_{m-1,m-1} H\beta).$$

Thus,

$$h_{m-1} = \beta^\top H I_{m-1}^\top (I_{m-1}\sigma\sigma^\top I_{m-1}^\top)^{-1} I_{m-1}(\mu - \eta_{m-1,m-1} H\beta).$$

Continuing this way we obtain additional values $\{h1, h_2, \ldots, h_{m-2}\}$ with

$$h_i = \beta^\top H I_i^\top (I_i\sigma\sigma^\top I_i^\top)^{-1} I_i(\mu - \eta_{i,i} H\beta)$$

such that for $h_i < h \le h_{i+1}$ only the constraint on the first i components of ζ bind, in which case

$$\nu_0^* = \frac{\beta^\top H I_i^\top (I_i\sigma\sigma^\top I_i^\top)^{-1} I_i\mu - h}{\beta^\top H I_i^\top (I_i\sigma\sigma^\top I_i^\top)^{-1} I_i H\beta},$$

ζ^* is given by (49) and λ is given by (48). Since $h \ge 0$ (see the proof of Proposition 4) and $h_1 = 0$, this characterization exhausts all possible cases.

The expressions for the optimal trading strategy immediately follow from the fact that for $h_i < h < h_{i+1}$

$$\begin{aligned} \theta(z,t) &= z\tilde{V}_{zz}(z,t)(\sigma\sigma^\top)^{-1}(\mu + \zeta^*) \\ &= \begin{cases} z\tilde{V}_{zz}(z,t)(\sigma\sigma^\top)^{-1}\mu & \text{if } h \ge h_{m+1} \\ z\tilde{V}_{zz}(z,t) I_i^\top (I_i\sigma\sigma^\top I_i^\top)^{-1} I_i(\mu - \nu_0^* H\beta) & \text{if } h_i < h \le h_{i+1}. \end{cases} \end{aligned} \tag{53}$$

The fact that $\operatorname{sign}(H\theta) \ge 0$ follows from the fact that (43) implies

$$H\theta = z\tilde{V}_{zz}\lambda \ge 0. \qquad \square$$

Proof of Corollary 2: Letting $\zeta = \nu_- - \nu_0$, it follows from equations (49) and (50) in the proof of Proposition 6 that, when $m = 1$ and $\mu > 0$,

$$\zeta^* = -\nu_0^*\beta \tag{54}$$

and

$$\nu_0^* = \left(\frac{\mu}{\beta} + \left(\frac{\sigma}{\beta} \right)^2 \frac{\tilde{V}_z + k\, VaR}{z \tilde{V}_{zz}} \right)^+ . \tag{55}$$

Substituting equations (54) and (55) in (22) yields the PDE in the Corollary. The expression for the optimal trading strategy follows directly from equations (53)–(55). □

Proof of Proposition 7: The expression for $\tilde{V}^P(z, K^-, VaR, t)$ in the last reporting period follows from direct integration, using the fact that for $t \in [T - \tau, T]$

$$V^P(z, K^-, VaR, t) = \mathrm{E}_t \left[\tilde{v} \left(z e^{-\frac{|\kappa|^2}{2}(T-t) - \kappa^\top (w(T) - w(t))}, K^-, VaR, T \right) \right],$$

where $\tilde{v}(z, K^-, VaR, T)$ is the function given explicitly in Appendix C.

In addition, it can be easily verified that under the assumptions of the Proposition the primal value function is homogeneous of degree $1 - \gamma$ in K^- when keeping the ratio of the other monetary quantities to K^- constant, i.e., that

$$V^P(K, K^-, VaR, t) = \frac{(K^-)^{1-\gamma}}{1-\gamma} V^P(K/K^-, 1, VaR/K^-, t).$$

This implies that

$$v(K, K^-, VaR, h\tau) = \alpha_h v(K, K^-, VaR, T)$$

where α_h is the constant in (34) and hence that

$$\tilde{v}(z, K^-, VaR, h\tau) = \alpha_h \tilde{v}(z/\alpha_h, K^-, VaR, T)$$

and

$$\begin{aligned} \tilde{V}^P(z, K^-, VaR, t) &= \mathrm{E}_t \left[\tilde{v} \left(z e^{-\frac{|\kappa|^2}{2}(h\tau - t) - \kappa^\top (w(h\tau) - w(t))}, K^-, VaR, h\tau \right) \right] \\ &= \alpha_h \tilde{V}^P(z/\alpha_h, K^-, VaR, T - (h\tau - t)) \end{aligned}$$

for $t \in [(h-1)\tau, h\tau)$. □

Appendix C: Dual Utility Function

In this appendix, we provide explicit expressions for the dual utility function in (19). Since these expression follow from straightforward but tedious algebra, they are given without proof.

To simplify notation, let

$$v(K) = v(K, K^-, \mathit{VaR}, i, k, h\tau)$$

and

$$\tilde{v}(z) = \tilde{v}(z, K^-, \mathit{VaR}, i, k, h\tau) = \max_{K \geq k\,\mathit{VaR}} \left[v(K) - zK \right]. \tag{56}$$

Also, let

$$\tilde{V}_1(z) = \max_{K \geq 0} \left[\max_{\mathit{VaR}_1 \geq 0} \left[\min_{z' \geq 0} \left[\tilde{V}(z', K, \mathit{VaR}_1, i_1, k_1, h\tau) + z'K \right] \right] - zK \right]$$

and

$$\tilde{V}_2(z) = \max_{K \geq 0} \left[\max_{\mathit{VaR}_2 \geq 0} \left[\min_{z' \geq 0} \left[\tilde{V}(z', K, \mathit{VaR}_2, i_2, k_2, h\tau) + z'K \right] \right] - zK \right],$$

where

$$i_1 = \begin{cases} 0 & \text{if } h\tau \in \mathcal{T}, \\ i & \text{otherwise }, \end{cases} \qquad i_2 = \begin{cases} 0 & \text{if } h\tau \in \mathcal{T}, \\ i+1 & \text{otherwise}, \end{cases}$$

$$k_1 = \begin{cases} k(i) & \text{if } h\tau \in \mathcal{T}, \\ k & \text{otherwise}, \end{cases} \qquad k_2 = \begin{cases} k(i+1) & \text{if } h\tau \in \mathcal{T}, \\ k & \text{otherwise}, \end{cases}$$

and

$$\tilde{V}(z', K_1, \mathit{VaR}_1, i_1, k_1, T) = \tilde{V}(z', K_2, \mathit{VaR}_2, i_2, k_2, T) = \max_{K \geq 0} \left[u(K) - z'K \right].$$

Finally, let z_1 be the unique solution of the equation

$$\tilde{V}_1'(z_1) + K^- - \mathit{VaR} = 0$$

and let z_2 be the unique solution of the equation

$$\tilde{V}_1(z_1) - \tilde{V}_2\left(\frac{z_2}{1+\lambda}\right) + (K^- - \mathit{VaR})\left(z_1 - \frac{z_2}{1+\lambda}\right) = 0$$

if

$$\tilde{V}_1(z_1) - \tilde{V}_2\left(\varepsilon^{-\gamma} e^{(r+\frac{\mu}{\beta})(T-h\tau)}\right) + (K^- - \mathit{VaR})\left(z_1 - \varepsilon^{-\gamma} e^{(r+\frac{\mu}{\beta})(T-h\tau)}\right) < 0 \tag{57}$$

and

$$z_2 = (1+\lambda)\varepsilon^{-\gamma} e^{(r+\frac{\mu}{\beta})(T-h\tau)}$$

otherwise.

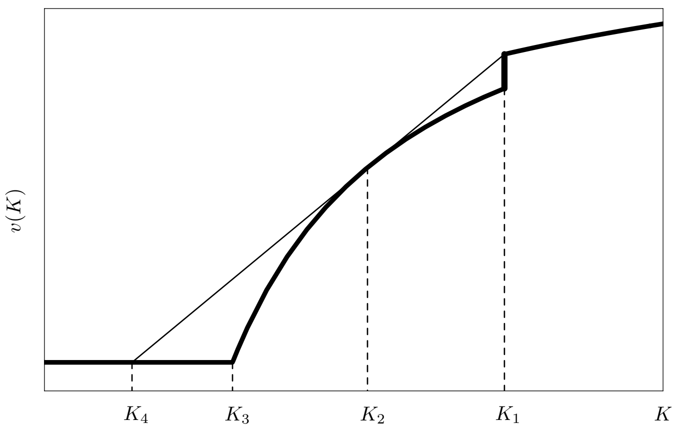

Figure 8: The graph illustrates the shape of the primal value function $v(K)$

Figure 8 shows the shape of $v(K)$, the primal value function at the end of the generic reporting period h. This function has a single discontinuity at $K_1 = K^- - VaR$ (the level of terminal capital below which a violation is recorded for the current reporting period), because an additional violation implies higher expected capital requirements in the future and hence a lower expected utility.[42] In addition, $v(K)$ is flat and equal to $u(0)$ to the left of the default point $K_3 = \lambda(K^- - VaR)/(1+\lambda)$ (see Remark 2), because of limited liability. Figure 8 also shows the location of the chord from $v(K_1)$ tangent to the arc $\{v(K) : K_3 \leq K < K_1\}$ at the point K_2. The condition in equation (57) is necessary and sufficient for the existence of such a tangency point.[43] Moreover, when this condition is satisfied, the constants z_1 and z_2 defined above are the slopes of $v(K)$ at the points K_1 and K_2, respectively.

The maximization of $v(K) - zK$ in the definition of $\tilde{v}(z)$ is made non-trivial by the non-concavity of $v(K)$ and by the constraint $K \geq k\,VaR$. In particular, letting K_4 denote the point at which the chord through $v(K_1)$ and $v(K_2)$ intersects $u(0)$ (as shown in Figure 8), four different cases are possible, depending on whether $k\,VaR$ is below K_4, between K_4 and K_3, between K_3 and K_2 or between K_2 and K_1 (the inequality in (17) implies that the case $VaR > K_1$ is irrelevant). This four cases are considered in turn below.

(i) If $0 \leq k\,VaR \leq K_4$, or equivalently if

$$0 \leq VaR \leq \frac{\frac{\varepsilon^{1-\gamma}}{1-\gamma} - (\tilde{V}_1(z_1) - K^-(z_2 - z_1))}{(1+k)z_2 - z_1},$$

[42] The discontinuity is absent if the reporting period is in the last backtesting period in the investment horizon.

[43] For simplicity, the graphs in this Appendix are drawn for the case in which the condition in in equation (57) is satisfied and $u(0)$ is finite. However, the given characterization of the dual value function remains valid even if these conditions are violated.

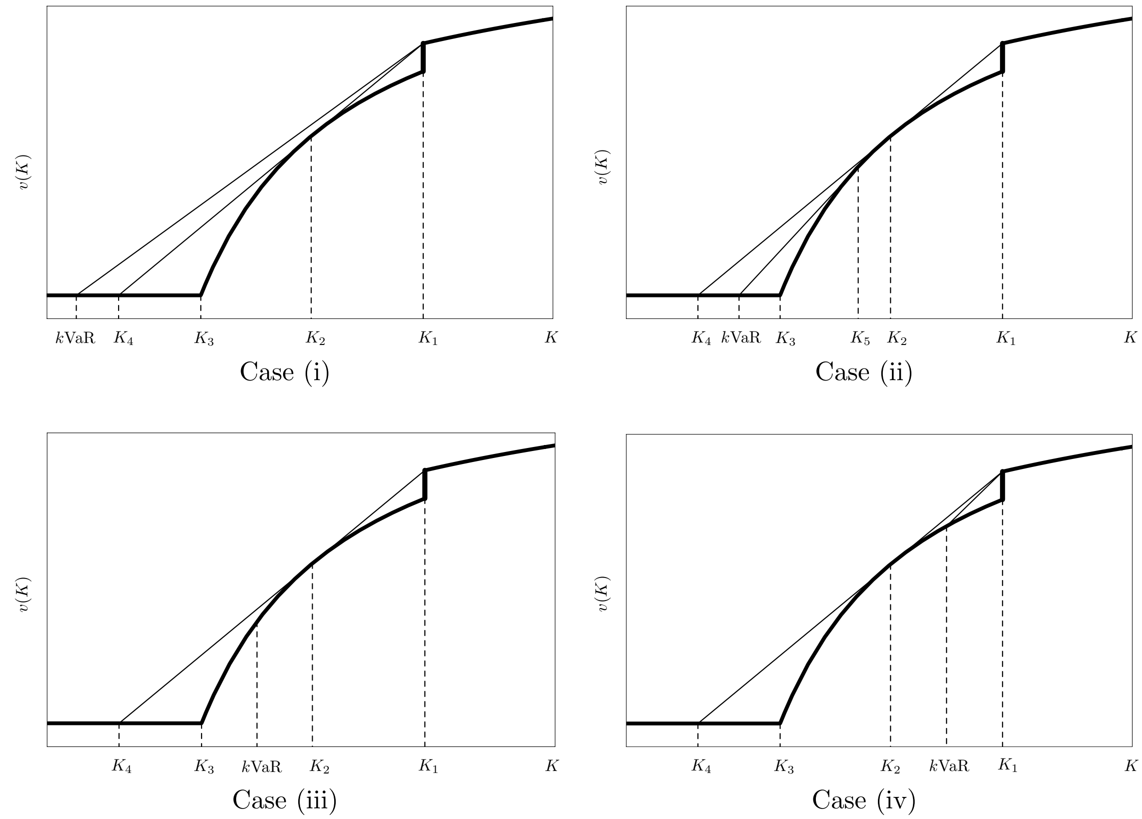

Figure 9: The graph illustrates the four possible cases in the construction of the dual value function $\tilde{v}(z)$.

then the solution of the maximization in (56) is given by $K = f(z)$, where

$$f(z) = \begin{cases} -\tilde{V}_1'(z) & \text{if } z < z_1 \\ K^- - VaR & \text{if } z_1 \leq z < z_3 \\ k\,VaR & \text{if } z \geq z_3, \end{cases}$$

and

$$z_3 = \frac{\tilde{V}_1(z_1) + (K^- - VaR)z_1 - \frac{\varepsilon^{1-\gamma}}{1-\gamma}}{K^- - (1+k)\,VaR}.$$

This implies

$$\tilde{v}(z) = \begin{cases} \tilde{V}_1(z) & \text{if } z < z_1 \\ \tilde{V}_1(z_1) - (K^- - VaR)(z - z_1) & \text{if } z_1 \leq z < z_3 \\ \frac{\varepsilon^{1-\gamma}}{1-\gamma} - k\,VaR\,z & \text{if } z \geq z_3. \end{cases}$$

This case is illustrated graphically in the first panel of Figure 9. The constant z_3 defined above is the slope of the chord from $v(k\,VaR)$ to $v(K_1)$. It should be easy to see that three subcases are possible. If $z < z_1$, the optimal value of K is to the right of K_1 and satisfies the first-order condition $v'(K) = z$. If $z_1 \leq z < z_3$, the optimal value of K equals K_1. Finally, if $z \geq z_3$, the optimal value of K coincides with the lower bound $k\,VaR$.

(ii) If $K_4 < k\,VaR \leq K_3$, or equivalently if

$$\frac{\frac{\varepsilon^{1-\gamma}}{1-\gamma} - (\tilde{V}_1(z_1) - K^-(z_2 - z_1))}{(1+k)z_2 - z_1} < VaR \leq \frac{\lambda K^-}{\lambda(1+k)+k},$$

then the solution of the maximization in (56) is given by $K = f(z)$, where

$$f(z) = \begin{cases} -\tilde{V}_1'(z) & \text{if } z < z_1 \\ K^- - VaR & \text{if } z_1 \leq z < z_2 \\ -\frac{1}{1+\lambda}\left(\tilde{V}_2(\frac{z}{1+\lambda}) - \lambda(K^- - VaR)\right) & \text{if } z_2 \leq z < z_3 \\ k\,VaR & \text{if } z \geq z_3, \end{cases}$$

$z_3 = +\infty$ if $u(0) = -\infty$ and z_3 solves

$$\tilde{V}_2\left(\frac{z_3}{1+\lambda}\right) - \frac{\lambda(K^- - VaR)}{1+\lambda}z_3 - \frac{\varepsilon^{1-\gamma}}{1-\gamma} + k\,VaR\,z_3 = 0$$

otherwise. This implies

$$\tilde{v}(z) = \begin{cases} \tilde{V}_1(z) & \text{if } z < z_1 \\ \tilde{V}_1(z_1) - (K^- - VaR)(z - z_1) & \text{if } z_1 \leq z < z_2 \\ \tilde{V}_2(\frac{z}{1+\lambda}) - \frac{\lambda(K^- - VaR)}{1+\lambda}z & \text{if } z_2 \leq z < z_3 \\ \frac{\varepsilon^{1-\gamma}}{1-\gamma} - k\,VaR\,z & \text{if } z \geq z_3. \end{cases}$$

This case is illustrated graphically in the second panel of Figure 9. Letting K_5 denote the point at which the chord from $v(k\,VaR)$ is tangent to the arc $\{v(K) : K_3 \leq K < K_1\}$, the constant z_3 defined above is equal to $v'(K_5)$. Four subcases are then possible. If $z < z_1$, the optimal value of K is to the right of K_1 and satisfies the first-order condition $v'(K) = z$. If $z_1 \leq z < z_2$, the optimal value of K equals K_1. If $z_2 \leq z < z_3$, the optimal value of K is between K_5 and K_2 and again satisfies the first-order condition $v'(K) = z$. Finally, if $z \geq z_3$, the optimal value of K coincides with the lower bound $k\,VaR$.

(iii) If $K_3 < k\,VaR \leq K_2$, or equivalently if

$$\frac{\lambda K^-}{\lambda(1+k)+k} < VaR \leq \frac{\lambda K^- - \tilde{V}_2'\left(\frac{z_2}{1+\lambda}\right)}{\lambda(1+k)+k},$$

then the solution of the maximization in (56) is given by $K = f(z)$, where

$$f(z) = \begin{cases} -\tilde{V}_1'(z) & \text{if } z < z_1 \\ K^- - VaR & \text{if } z_1 \leq z < z_2 \\ -\frac{1}{1+\lambda}\left(\tilde{V}_2(\frac{z}{1+\lambda}) - \lambda(K^- - VaR)\right) & \text{if } z_2 \leq z < z_3 \\ k\,VaR & \text{if } z \geq z_3, \end{cases}$$

and z_3 solves

$$\frac{1}{1+\lambda}\left(\tilde{V}_2'\left(\frac{z_3}{1+\lambda}\right) - \lambda(K^- - VaR)\right) + k\,VaR = 0.$$

This implies

$$\tilde{v}(z) = \begin{cases} \tilde{V}_1(z) & \text{if } z < z_1 \\ \tilde{V}_1(z_1) - (K^- - \ VaR)(z - z_1) & \text{if } z_1 \leq z < z_2 \\ \tilde{V}_2(\frac{z}{1+\lambda}) - \frac{\lambda(K^- - VaR)}{1+\lambda} z & \text{if } z_2 \leq z < z_3 \\ \tilde{V}_2(\frac{z_3}{1+\lambda}) - \frac{\lambda(K^- - VaR)}{1+\lambda} z_3 - k\, VaR\ (z - z_3) & \text{if } z \geq z_3. \end{cases}$$

This case is illustrated graphically in the third panel of Figure 9 and is identical to the previous one, except that z_3 (the value of z above which the optimal value of K coincides with $k\, VaR$) is now given by $v'(k\, VaR)$.

(iv) If $K_2 < k\, VaR \leq K_1$, or equivalently if

$$\frac{\lambda K^- - \tilde{V}_2'\left(\frac{z_2}{1+\lambda}\right)}{\lambda(1+k)+k} < VaR \leq \frac{K^-}{1+k},$$

then the solution of the maximization in (56) is given by $K = f(z)$, where

$$f(z) = \begin{cases} -\tilde{V}_1'(z) & \text{if } z < z_1 \\ K^- - \ VaR & \text{if } z_1 \leq z < z_3 \\ k\, VaR & \text{if } z \geq z_3, \end{cases}$$

$$z_3 = \frac{\tilde{V}_1(z_1) + (K^- - \ VaR)z_1 - (\tilde{V}_2(z_4) + ((1+\lambda)k\, VaR - \lambda(K^- - \ VaR))z_4)}{K^- - (1+k)\, VaR}.$$

and z_4 solves

$$\tilde{V}_2'(z_4) + (1+\lambda)k\, VaR - \lambda(K^- - \ VaR) = 0.$$

This implies

$$\tilde{v}(z) = \begin{cases} \tilde{V}_1(z) & \text{if } z < z_1 \\ \tilde{V}_1(z_1) - (K^- - \ VaR)(z - z_1) & \text{if } z_1 \leq z < z_3 \\ \tilde{V}_1(z_1) - (K^- - \ VaR)(z_3 - z_1) - k\, VaR(z - z3)z & \text{if } z \geq z_3. \end{cases}$$

This case is illustrated graphically in the fourth panel of Figure 9. The constants z_3 and z_4 defined above are,respectively, the slope of the chord from $v(k\, VaR)$ to $v(K_1)$ and the slope of $v(K)$ at $K = k\, VaR$. □

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